EFFECTS OF SUCTION AND INJECTION ON SELF-SIMILAR SOLUTIONS OF SECOND-ORDER BOUNDARY-LAYER EQUATIONS

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CERTIFICATE

Certified that this work on "EFFECTS OF SUCTION AND INJECTION ON SECOND ORDER SELF-SIMILAR SOLUTIONS OF BOUNDARY LAYER FLOWS" by Shabbir Ahmad has been carried out under my supervision and that this has not been submitted elsewhere for a degree.

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CONTENTS

SYNOPSIS

Effect of suction or injection on self-similar solutions on second order (due to longitudinal curvature, transverse curvature, displacement speed, external verticity and temperature gradient) boundary layer flows has been studied. Two cases of prescribed wall temperature and that of insulated wall with full similarity in viscous dissipation are considered. Numerical solutions have been obtained for large values of parameters and the results are critically discussed.

CHAPTER 1

INTRODUCTION

The problem of boundary layer flows with suction or injection is of great importance in engineering applications, like boundary layer control, transpiration cooling and other diffusional operations. It has been studied extensively and references thereof can be found in the texts by Schlichting (1968), Eckert (1972) and Rosenhead (1963). These studies have employed the classical boundary layer equations and hence are valid for very large values of Reynolds numbers. At moderately large Reynolds number, however, it is wellknown that the boundary layer equations require certain second order corrections, (VanDyke 1962) whose order of magnitude is $R^{-\frac{1}{2}}$ when compared to Prandtle's boundary layer. These corrections may be attributed to arise from longitudinal curvature, transverse curvature, displacement speed, external vorticity and stagnation enthalpy gradient.

In the present work we study the effect of suction or injection on these second order corrections. To this end we have studied the effect of suction and injection on self-similar solutions of second order momentum and energy equations with full similarity in viscous dissipation.

For an impermeable wall the general structure of self similarity for second order effects have been studied by Afzal and Oberai (1972) and Warle and Davis (1970). These authors have

shown that in addition to Falkner-Skan pressure gradient parameter β, the second order equations contain three new parameters Λ_ℓ, due to longitudinal curvature, Λ_t due to transverse curvature and Λ_d due to displacement effect. Further it is shown that for a given β, the second order solutions show a large number of singularities for certain critical values of Λ_ℓ, Λ_t and Λ_d. Physical as well as mathematical structure of these singularities is explained by Raisinghani (1973). In particular it is shown that the singularities occur whenever the parameters Λ_ℓ etc. assume a value equal to eigen value of the corresponding homogeneous problem with homogeneous boundary conditions.

Generalization to heat transfer with full similarity in viscous dissipation has been made by Raisinghani (1973) and Afzal and Raisinghani (1973). They have studied the two cases of prescribed wall temperature and of insulated wall. For the latter case the classical concept of recovery factor has been extended to second order flows. For longitudinal curvature problem, the jointly self-similar solutions for second order momentum equations have been studied by Narsimha and Ojha (1967), while the corresponding heat transfer problem for a prescribed wall temperature has been studied by Kadambi and Gupta (1971).

For our study, the self-similar solutions with suction and injection have been obtained following the analysis of Raisinghani (1973), which involve an additional suction parameter C.

($C > 0$ represents the suction while $C < 0$, the injection). The equations have been integrated numerically for various values of C in the range of $-.6 \leq C \leq .6$. To see clearly the effect of suction and injection, solutions are obtained for $\beta = 0$, $.5$ and 1, (as they represent important cases of flat plate, plane stagnation and axisymmetric stagnation flows), Prandtl number $\sigma = .7$, 3 and 5, and for a large values of Λ_ℓ etc. in the range of $-7.5 \leq \Lambda_\ell$, Λ_t, $\Lambda_d \leq 4$.

The transverse curvature problem with suction and injection has been studied earlier by Wannous and Sparrow (1965). These authors have studied a particular problem of momentum transfer in axial flow along a circular cylinder and have shown that suction reduces the second order skin friction while injection accentuates it. In turms of our parameters their analysis corresponds to $\beta = 0$ and $\Lambda_t = 0$.

CHAPTER 2

FIRST AND SECOND ORDER BOUNDARY LAYER EQUATIONS

2.1 Introduction

The second order boundary layer equations are obtained from Navier-Stokes equations using the method of matched asymptotic expansions, VanDyke (1962). In this method the flow field is divided into two regions one near the wall of the order of $R^{-\frac{1}{2}}$ and the other outside this region. The two regions are described by two separate asymptotic expansions with $R^{-\frac{1}{2}}$ as an expansion parameter. The two expansions are matched in the overlap region.

2.2 Formulation Of Second Order Problem

The Navier-Stokes equations in a non-dimensional vector form (here distance $\underset{\sim}{X}(s,n)$, velocity $\underset{\sim}{U}(U,V)$ are non dimentionalized with respect to characteristic length L of the body and reference velocity U_c respectively, pressure by ρU_c^2 and temperature by U_c^2/c_p) may be written as

$$\text{div } \underset{\sim}{U} = 0 \; , \tag{2.1}$$

$$\underset{\sim}{U} \cdot \text{grad } \underset{\sim}{U} + \text{grad } P = -R^{-1} \text{ curl curl } \underset{\sim}{U} \tag{2.2}$$

$$\underset{\sim}{U} \cdot \text{grad } T - \sigma^{-1} R^{-1} \nabla^2 T = R^{-1} \text{ grad } \underset{\sim}{U} \cdot \text{def } \underset{\sim}{U}, \tag{2.3}$$

where $R = U_c L \rho/\mu$ is the characteristic Reynolds number, $\sigma = \mu c_p/k$, the Prandtle number. In writing the above equations, the flow is assumed incompressible.

The boundary conditions at the surface are

$$U = 0 \quad , \; V = V_w(s) \tag{2.4}$$

$$T = T_w(s) \quad \text{or grad } T = 0 \tag{2.5}$$

where $V_w(s)$ is of the order of $R^{-\frac{1}{2}}$. Far upstream the flow has to approach prescribed, may be non-uniform, velocity and temperature fields.

The outer asymptotic expansions in the limit $\underset{\sim}{X}(s,n)$ fixed $R \to \infty$ are

$$U = U_1(s,n) + R^{-\frac{1}{2}} U_2(s,n) + \ldots \tag{2.6a}$$

$$V = V_1(s,n) + R^{-\frac{1}{2}} V_2(s,n) + \ldots \tag{2.6b}$$

$$P = P_1(s,n) + R^{-\frac{1}{2}} P_2(s,n) + \ldots \tag{2.6c}$$

$$T = T_1(s,n) + R^{-\frac{1}{2}} T_2(s,n) + \ldots \tag{2.6d}$$

Substituting these expansions in (2.1-2.3) and collecting terms of equal powers of R we get the well-known Euler's equations whose integration along stream line gives

$$P_1 + U_1^2/2 = B_1(\Psi_1) \tag{2.7a}$$

$$T_1 = H_1(\Psi_1) \tag{2.7b}$$

where Ψ_1 is the first order outer stream function and $B_1(\Psi_1)$ and $H_1(\Psi_1)$ are Bernoullis functions to be determined for upstream conditions.

The second order equations are perturbation form of Euler equations and their integration gives

$$P_2 + \underset{\sim}{U}_1 \cdot \underset{\sim}{U}_2 = \Psi_2 \, B_1' \, (\Psi_1) + B_2(\Psi_1) \tag{2.8a}$$

$$T_2 = \Psi_2 \, H_1' \, (\Psi_1) + H_2(\Psi_1) \tag{2.8b}$$

If the on-coming stream is independent of Reynolds number, then

$$B_2 = 0 = H_2$$

The outer equations are invalid in a region of order $R^{-\frac{1}{2}}$ near the wall. To study flow near the wall, the appropriate (Prandtle) variable

$$N = n \, R^{\frac{1}{2}} \tag{2.9}$$

and the corresponding inner limit N fixed, $R \to \infty$ gives the following inner asymptotic expansions

$$U = u_1(s,N) + R^{-\frac{1}{2}} u_2(s,N) + \dots \tag{2.10a}$$

$$V = R^{-\frac{1}{2}} v_1(s,N) + R^{-1} v_2(s,N) + \dots \tag{2.10b}$$

$$P = p_1(s,N) + R^{-\frac{1}{2}} p_2(s,N) + \dots \tag{2.10c}$$

$$T = t_1(s,N) + R^{-\frac{1}{2}} t_2(s,N) + \dots \tag{2.10d}$$

Using these expansions in equations (2.1-2.3), we get first and second order governing equations for inner (boundary layer) flow.

These inner equations are valid in a region of order $R^{-\frac{1}{2}}$ and violate the conditions at infinity. Matching the two solutions we

obtain the missing boundary conditions needed to solve the boundary layer equations.

2.2.1 First Order Boundary Layer Equations

Continuity

$$(r^j u_1)_s + (r^j v_1)_N = 0 , \qquad (2.11)$$

Momentum

$$u_1 u_{1s} + v_1 u_{1N} - U_1(s,0)\, U_{1s}(s,0) - u_{1NN} = 0, \qquad (2.12)$$

Energy

$$u_1\ t_{1N} + v_1\ t_{1N} - \sigma^{-1}\ t_{1NN} - u_{1N}^2 = 0 . \qquad (2.13)$$

Boundary and Matching conditions are

$$\left.\begin{aligned} &u_1(s,0) = 0 \ , \ v_1(s,0) = V_w(s) \\ &u_1(s,N) = U_1(s,0) \quad \text{as } N \to \infty \end{aligned}\right\} \qquad (2.14)$$

$$\left.\begin{aligned} &t_1(s,0) = t_w(s) \text{ or } t_{1N}(s,0) = 0 \\ &t_1(s,N) = H_1(0) \quad \text{as } N \to \infty . \end{aligned}\right\} \qquad (2.15)$$

2.2.2 Second Order Boundary Layer Equations

Continuity

$$\begin{aligned}(r^j u_2)_s + (r^j v_2)_N = &- [\, r^j (j \cos\theta/r)\, N\, u_1\,]_s \\ &- [\, r^j\ (K + j \cos\theta/r)\, N\, v_1\,]_N , \qquad (2.16)\end{aligned}$$

Momentum

$$u_1 u_{2s} + u_2 u_{1s} + v_1 u_{2N} + v_2 u_{1N} - u_{2NN}$$

$$= [N u_1 u_{1s} - u_1 v_1 - N U_1(s,0) U_{1s}(s,0) + u_{1N}]$$

$$- [K N U_1^2 (s,0) + K \int_N^\infty \{U_1^2 (s,0) - u_1^2\} dN]_s + u_{1N} j \cos \theta/r$$

$$- r^j B_1(0) V_2(s,0) + [U_1(s,0) U_2(s,0)]_s \qquad (2.17)$$

Energy

$$u_1 t_{2s} + u_2 t_{1s} + v_1 t_{2N} + v_2 t_{1N} - \sigma^{-1} t_{2NN} - 2u_{1N} u_{2N}$$

$$= K (N u_1 t_{1s} + \sigma^{-1} t_{1N} - 2u_1 u_{1N}) + \sigma^{-1} j \cos \theta \, t_{1N}/r \quad (2.18)$$

Boundary and matching conditions are

$$u_2(s,0) = 0 , \; v_2(s,0) = 0$$

$$u_2(s,N) = N K U_1(s,0) + r^j N B_1'(0) + U_2(s,0) \text{ as } N \to \infty \qquad (2.19)$$

$$t_2(s,0) = 0 \text{ or } t_{2N}(s,0) = 0$$

$$t_2(s,N) = H_1'(0) \, \psi_1(s,N) \quad N \to \infty \qquad (2.20)$$

where $H_1'(0) = (\frac{\partial H_1}{\partial \psi_1})_{\psi_1=0}$

The second order equations are linear and hence can be divided into a number of simpler problems. The terms proportional to K, $j \cos \theta/r$, $U_2(s,0)$, $B_1'(0)$ and $H_1'(0)$ may be attributed to arise due to longitudinal curvature, transverse curvature, displacement

speed, external vorticity and temperature gradient respectively.

In the following pages quantities $U_1(s,0)$, $U_2(s,0)$, $H_1(0)$, $B_1(0)$ will be written without their arguments as U_1, U_2, H_1 and B_1.

CHAPTER 3

SELF-SIMILAR ANALYSIS

3.1 Introduction

As mentioned earlier, self-similar solutions of first and second order boundary layer equations have been obtained by Afzal and Raisinghani (1973) for flow over an impermeable surface ($V_w=0$). Here we shall study self similar solutions of the first and second order problem in the presence of suction and injection at the wall, for an incompressible two dimensional and axisymmetric flows.

The governing equations for first and second order problem are transformed using the Gortlers variables

$$\xi = \int_0^s U_1 \, r^{2j} \, ds \, , \quad \eta = \frac{r^j \, N \, U_1}{\sqrt{2\xi}} \tag{3.1}$$

The equations obtained in terms of ξ and η are subjected to the condition of similarity and finally equations with full similarity in dissipation are obtained.

3.2 Governing Equations For Non-Similar Flows

3.2.1 First Order Problem

The first order stream function ψ_1 defined by equation of continuity (2.9)

$$\psi_{1N} = r^j \, u_1 \quad , \quad \psi_{1s} = -r^j \, v_1 \, . \tag{3.2}$$

The stream function ψ_1 and temperature t_1 are expressed as

$$\psi_1(s,N) = \sqrt{2\xi}\ \ f(\xi,\eta) \ , \tag{3.3}$$

$$t_1(s,N) = (t_w - H_1)\ g(\xi,\eta) + H_1. \tag{3.4}$$

Using (3.2) and (3.3) we get for the velocity components

$$u_1 = U_1\ f' \tag{3.5}$$

$$v_1 = -U_1\ r^j[f + 2\xi f_\xi + (\Lambda_{U_1} + j\ \Lambda_{r}-1)\ \eta f']/\sqrt{2\xi} \tag{3.6}$$

where Λ_r is defined by

$$\Lambda_\phi = \frac{2\xi}{\phi}\ \frac{d\phi}{d\xi} \tag{3.7}$$

The governing equations (2.12 - 2.15) may be written as

momentum

$$f''' + ff'' + \Lambda_{U_1}(1-f'^2) - 2\xi(f'f'_\xi - f_\xi f'') = 0, \tag{3.8}$$

Energy equation

$$\sigma^{-1}\ g'' + f\ g' - \Lambda_{t_w-H_1} f'\ g - 2\xi(f'\ g_\xi - f_\xi\ g') = -\ E\ f''^2, \tag{3.9}$$

Boundary conditions

$$f(\xi,0) + 2\xi\ f_\xi(\xi,0) = C(\xi) \tag{3.10a}$$

$$f'(\xi,0) = 0\ ,\quad f'(\xi,\infty) = 1 \tag{3.10b}$$

$$g(\xi,0) = 1\ ,\quad g(\xi,\infty) = 0\ . \tag{3.11}$$

Where $C(\xi) = -V_w\ \sqrt{2\xi}/(U_1\ r^j)$ is the suction parameter and $E = U_1^2/(t_w - H_1)$ is the Eckert number, suffix ξ and dash represents differentiation with respect to ξ and η respectively. Λ_{U_1} and $\Lambda_{t_w-H_1}$ are defined by (3.7) and are known as principal velocity function

principal thermal function respectively. These principal functions are of fundamental importance in describing flow conditions. In non-similar flows, the principal functions vary from point to point while they remain constant along a stream line in case of similar flows. Moreover if the two flows are similar, the value of principal functions remain the same for the corresponding stream lines.

(3.2.2) Second Order Problem

The second order stream function ψ_2, satisfying equation of continuity (2.14), is defined as

$$\psi_{2N} = r^j[u_2 + j\ N\ u_1 \cos\theta/r],$$
$$\psi_{2s} = -r^j\ [v_2 + (K + j\cos\theta/r)\ N\ v_1]. \qquad (3.12)$$

In terms of variables ξ, η the stream function ψ_2 and temperature t_2 are expressed as

$$\psi_2 = \sqrt{2\xi}\ F(\xi,\eta) \qquad (3.13)$$

$$t_2 = (t_w - H_1)\ G(\xi,\eta) \qquad (3.14)$$

The velocity components obtained from (3.12) and (3.13) are

$$u_2 = U_1\ [F'(\xi,\eta) - B_t\ \eta\ f'] \qquad (3.15)$$

$$v_2 = U_1\ r^j\ [-\{F + 2\xi\ F_\xi + \eta(\Lambda_{U_1} + j\ \Lambda_r - 1)\ F'\}$$
$$+ \eta(B_\ell + B_t)\ \{f + 2\xi\ f_\xi + \eta(\Lambda_{U_1} + j\ \Lambda_r - 1)f'\}]/\sqrt{2\xi} \qquad (3.16)$$

where B_ℓ and B_t are defined by (3.22) and (3.23). Using (3.12-3.16) and the second order governing equations (2.15-2.18), we have

Momentum equation

$$F''' + fF'' - 2\Lambda_{U_1} f'F' + f''F - 2\xi\,(f'_\xi F' - f'' F_\xi + f'F'_\xi - f_\xi F'')$$

$$= B_\ell[\{-\eta(1+\Lambda_{U_1})\ f''' + (\Lambda_{B_\ell} + \Lambda_{U_1} - 1)\ (f'' + ff' + 2\xi\ f'\ f_\xi$$

$$+ 4\xi \int_\eta^\infty f'_\xi\ f'\ d\eta) + (\Lambda_{B_\ell} + 2\Lambda_{U_1})\ (\eta\ \Lambda_{U_1} + \alpha + 2\xi\ \alpha_\xi)\}$$

$$/(1 + \Lambda_{U_1})] + B_t[-\eta(2\Lambda_{U_1} + f''') + f'' + ff' - \Lambda_{B_t}\ \eta\ f'^2$$

$$+ 2\xi\ f_\xi\ f'] - B_v(\alpha + 2\xi\ \alpha_\xi) - B_d(2\Lambda_{U_1} + \Lambda_{B_d}) \tag{3.17}$$

Energy equation

$$\sigma^{-1}\ G'' + f\ G' - \Lambda_{t_w-H_1} f'\ G - 2\xi\ (f'\ G_\xi - f_\xi\ G')$$

$$= \Lambda_{t_w-H_1}\ F'\ g - F\ g' - 2\xi(F'\ g_\xi - F_\xi\ g') - 2E\ f''\ F''$$

$$+ B_\ell[-\sigma^{-1}\ (\eta\ g')' + E\ f''\ (-\eta\ f'' + 2f')] + B_t[-\sigma^{-1}(\eta\ g')'$$

$$+ E\ f''\ (\eta\ f'' + 2f')] \tag{3.18}$$

Boundary conditions

$$F(\xi,0) + 2\xi\ F_\xi(\xi,0) = 0 \tag{3.19a}$$

$$F'(\xi,0) = 0 \tag{3.19b}$$

$$F'(\xi,\eta) = \eta(-B_\ell + B_t + B_v) + B_d \quad \eta \to \infty \tag{3.19c}$$

$$G(\xi,0) = 0\ ,\ G'(\xi,0) = 0 \tag{3.20a}$$

$$G(\xi,\eta) = B_e(\eta-\alpha) \text{ as } \eta \to \infty$$

where various quantities are defined by

$$\alpha = \lim_{\eta \to \infty} (\eta - f), \tag{3.21}$$

$$B_\ell = \sqrt{2\xi}\, K/(r^j\, U_1), \quad B_t = \sqrt{2\xi}\, j \cos\theta/(r^{2j}\, U_1) \tag{3.22-23}$$

$$B_d = U_2/U_1 \;, \; B_v = \sqrt{2\xi}\, B_1'/U_1^2 \tag{3.24-25}$$

$$B_e = \sqrt{2\xi}\, H_1'/(t_w - H_1), \tag{3.26}$$

and Λ_{B_ℓ} , Λ_{B_t} , Λ_{B_d} are principal functions of longitudinal curvature, transverse curvature, and displacement speed respectively and are defined by (3.7).

3.3 Self Similar Solutions :

3.3.1 First Order Problem

To study self-similar solutions of first order boundary layer equations, we assume

$$f = f(\eta) \;, \tag{3.27}$$

and Λ_{U_1} to be a constant say β . Now the equations (3.8) and (3.10) reduce to

$$f''' + ff'' + \beta(1-f'^2) = 0 \tag{3.28a}$$

$$f(0) = C$$

$$f'(0) = 0 \;, \; f'(\infty) = 1 \tag{3.28b}$$

where C, the suction parameter is to be constant.

Linearity of first order energy equation permits its study independently of Eckert number, i.e.,

$$g(\xi,\eta) = g_1(\xi,\eta) + E\, g_2(\xi,\eta)$$

If g_1 and g_2 are similar, they must be functions of η only. But still E may be a function of ξ, which renders g non-similar. Hence for full similarity with viscous dissipation E has to be constant, which implies $\Lambda_{t_w} - H_1 = 2\beta$, a constant. Therefore for full similarity, we assume

$$g(\eta) = g_1(\eta) + E\, g_2(\eta) \tag{3.29}$$

substituting (3.29) into (3.9) and collecting terms proportional to powers of E, we get

$$\sigma^{-1} g_1'' + f g_1' - 2\beta f' g_1 = 0 , \tag{3.30a}$$

$$g_1(0) = 1, \quad g_1(\infty) = 0 , \tag{3.30b}$$

$$\sigma^{-1} g_2'' + f g_2' - 2\beta f' g_2 + f''^2 = 0 , \tag{3.31a}$$

$$g_2(0) = 0 = g_2(\infty). \tag{3.31b}$$

3.3.2 Second Order Problem

The second order boundary layer equations are also linear and therefore allow superposition. These higher order equations can be divided into a number of simpler problems. In the equations (3.17) and (3.18), the terms proportional to B_ℓ arise due to longitudinal curvature, B_t due to transverse curvature, B_d due to displacement, B_v due to external verticity and B_e due to temperature gradient in the on-coming stream.

Further the second order heat transfer problem like that of first order can also be solved independent of Eckert number.

Therefore to obtain the self-similar equations for the second order problem we assume

$$F = \sum_{m=\ell,t,d,v} B_m F^{(m)}(\eta) , \tag{3.32}$$

$$G = \sum_{m=\ell,t,d,v,e} B_m [G_1^{(m)}(\eta) + EG_2^{(m)}(\eta)] \tag{3.33}$$

and Λ_{B_ℓ}, Λ_{B_t}, Λ_{B_d} are constants say Λ_ℓ, Λ_t and Λ_d respectively. Using equations (3.32-3.33) and separating the various second order effects, the equations (3.17-3.20) take the form (using operator notation)

$$F''' + f F'' - (2\beta + \lambda_i) f' F' + (1 + \lambda_i) f''F = L_i \tag{3.34a}$$

$$F(0) = 0 = F'(0) , F(\infty) = S_i \tag{3.34b}$$

$$\sigma^{-1}G_1'' + fG_1' - (2\beta+\lambda_i)f'G_1 + (1+\lambda_i)Fg_1' - 2\beta F' g_1 = M_i \tag{3.35a}$$

$$G_1(0) = 0 , G_1(\infty) = d_i \tag{3.35b}$$

$$\sigma^{-1}G_2'' + fG_2' - (2\beta+\lambda_i)f'G_2 + (1+\lambda_i)Fg_2'-2\beta F'g_2 = -2f''F'' + N_i \tag{3.36a}$$

$$G_2(0) = 0 = G_2(\infty) \tag{3.36b}$$

where λ_i, L_i, S_i, M_i, d_i and N_i are given in table 1. In the above equations the superscript i on F, G_1 and G_2 have been omitted for convinience of writing.

3.4 Skin Friction and Displacement Thickness

The skin friction in non-dimensional form may be written as

Table - 1

	i	λ_i	L_i	S_i	M_i	d_i	N_i
Longitudenal curvature	ℓ	Λ_ℓ	$-\eta f''' + [(\Lambda_\ell+\beta-1)(f''+ff') + (2\beta+\Lambda_\ell)(\beta\eta+\alpha)]/(1+\beta)$	$-\eta$	$-\sigma^{-1}(\eta\, g_1')'$	0	$-\sigma^{-1}(\eta g_2')'+f''(-\eta f''+2f')$
Transverse curvature	t	Λ_t	$-\eta(2\beta+f''') + f''+ff'-\Lambda_t\,\eta\, f'^2$	η	$-\sigma^{-1}(\eta g_1')'$	0	$-\sigma^{1}(\eta g_2')'+f''(\eta f''+2f'')$
Displacement speed	d	Λ_d	$-2\beta - \Lambda_d$	-1	0	0	0
Vorticity interaction	v	$1-2\beta$	$-\alpha$	η	0	0	0
Temperature gradient in on coming stream	e	$1-2\beta$	0	0	0	$\eta-\alpha$	0

$$\frac{\tau}{\rho\ U_c^2} = R^{-1/2} \left.\frac{\partial U}{\partial N}\right|_{N=0}$$

Making use of (2.11a), (3.5) and (3.15) we have for skin friction

$$\frac{\tau}{\rho\ U_c^2} = \frac{r^j\ U_1^2\ R^{-1/2}}{\sqrt{2\xi}}\ [f''(0) + R^{-1/2} \sum_{m=\ell,t,d,v} B_m F^{(m)''}(0) + \ldots] \tag{3.37}$$

To obtain the expressions for the first order and second order displacement thickness, we start with the basic definition of the displacement thickness δ^* defined as

$$\int_{\delta}^{n_o} U\ r_o^j\ dn = \int_0^{n_o} u\ r_o^j\ dn \tag{3.38}$$

where n_o is taken as the outer limit of inner boundary layer. The inner expansion (3.38) may be written as

$$\int_0^{\delta_1+R^{-1/2}\delta_2+\ldots} U(s,R^{-1/2}N)\, r_o^j dN = \int_0^{\infty} [U(s,R^{-1/2}N) - u(s,N)]\, r_o^j\ dN \tag{3.39}$$

The displacement thickness itself is expanded as

$$\delta^*(s,R^{-1/2}) = R^{-1/2}\ \delta_1(s) + R^{-1}\ \delta_2(s) + \ldots$$

Substituting outer expansions for $U(s,R^{\frac{1}{2}}N)$ and inner expansions for $u(s,N)$ and r_o^j, we have

$$\delta_1 = \int_0^{\infty} [1 - \frac{u_1(s,N)}{U_1(s,0)}]\ dN \tag{3.40}$$

and

$$\delta_2 = \frac{U_2(s,0)}{U_1(s,0)} \int_0^{\infty} [\frac{u_1(s,N)}{U_1(s,0)} - \frac{u_2(s,N) - N u_{1N}(s,0)}{U_2(s,0)}]\ dN$$
$$- \frac{1}{2}\frac{U_{1N}(s,0)}{U_1(s,0)}\,\delta_1^2 - \frac{J\cos\theta}{r^j}\,[\frac{\delta_1^2}{2} - \int_0^{\infty} N\ \{1 - \frac{u_1(s,N)}{U_1(s,0)}\}\ dN]$$

Substituting the similarity variables, the expression for the displacement thickness is

$$\delta^* = R^{-1/2} \frac{\sqrt{2\xi}}{r^j U_1} \delta + R^{-1} \left[\frac{\sqrt{2\xi}}{r^j U_1} \frac{U_2}{U_1} \delta^{(d)} + \frac{2\xi}{r^{2j} U_1^2} \left(\frac{r^j B_1'}{r U_1} \delta^{(v)} + K \delta^{(\ell)} + \frac{j \cos\theta}{r} \delta^{(t)}\right)\right] \tag{3.42}$$

where

$$\delta = \lim_{\eta \to \infty} (\eta - f) \tag{3.43a}$$

$$\delta^{(\ell)} = \lim_{\eta \to \infty} \left(\frac{1}{2}\eta^2 - F^{(\ell)}(\eta)\right) + \frac{\delta^2}{2} \tag{3.43b}$$

$$\delta^{(t)} = \lim_{\eta \to \infty} \left(\frac{1}{2}\eta^2 - F^{(t)}(\eta)\right) - \frac{\delta^2}{2} \tag{3.43c}$$

$$\delta^{(d)} = \lim_{\eta \to \infty} [\eta - F^{(d)}(\eta)] - \delta \tag{3.43d}$$

$$\delta^{(v)} = \lim_{\eta \to \infty} \left[\frac{1}{2}\eta^2 - F^{(v)}(\eta)\right] - \frac{\delta^2}{2} \tag{3.43e}$$

3.5 Heat Transfer and Recovery Factor

The total heat transfer rate at the wall in non-dimensional form

$$\frac{q}{\rho\, c_p\, T_c\, U_c} = -\sigma^{-1} R^{-1/2} \left.\frac{\partial T}{\partial N}\right|_{N=0}$$

Using equations (2.10), (3.4) and (3.14) we get

$$\frac{q}{\rho\, c_p\, T_c\, U_c} = -\sigma^{-1} r^j U_1(s,0)(t_w - H_1) R^{-1/2} [g_1'(0) + R^{-1/2} G'(0) + \ldots]/\sqrt{2\xi}$$

$$= -\sigma^{-1} r^j U_1(s,0)(t_w - H_1) R^{-1/2} [q_1 + R^{-1/2} q_2 + \ldots]/\sqrt{2\xi} \tag{3.44}$$

where $$q_1 = g_1'(0) + E\, g_2'(0) \tag{3.45}$$

and $$q_2 = \sum_{m=\ell,t,d,v,e} B_m[G_1^{(m)}(0) + E\, G_2^{(m)'}(0)] \tag{3.46}$$

The temperature of the wall for which there is no heat transfer at the surface is known as recovery temperature t_r. From equation [3.40] with $q = 0$, and from the expansion for recovery temperature

$$T_r = t_{r_1} + R^{-\frac{1}{2}} t_{r_2} + \ldots$$

where t_{r_1} and t_{r_2} are first and second order recovery temperature, we have expressions for t_{r_1} and t_{r_2} as

$$\frac{t_{r_1} - H_1}{U_1^2} = -\frac{g_2'(0)}{g_1'(0)} \tag{3.43}$$

and

$$\frac{t_{r_2}}{U_1^2} = \sum_{m=\ell,t,d,v} B_m \frac{[g_2'(0)G_1^{(m)'}(0) - g_1'(0)G_2^{(m)'}(0)]}{g_1'^2(0)} + \bar{B}_e \frac{G_1^{(e)'}(0)}{g_1'(0)} \tag{3.44}$$

where $$\bar{B}_e = -H_1' \sqrt{2\xi}/U_1^2 \tag{3.45}$$

Recovery temperature is generally expressed in terms of dimensionless recovery factor defined as

$$\begin{aligned} r_f &= 2(t_r - H_1)/U_1^2 \\ &= r_1 + R^{-\frac{1}{2}} r_2 + \ldots \end{aligned} \tag{3.46}$$

where r_1 and r_2 are the first and second order recovery factors

and are given by

$$r_1 = -\,2g_2'(0)/g_1'(0) \tag{3.47}$$

$$r_2 = \sum_{m=\ell,t,d,v} \{2\, B_m[g_2'(0)G_1^{(m)'}(0) - g_1'(0)\; G_2^{(m)'}(0)]/g_1'^{\,2}(0)\}$$

$$+\; 2\;\bar{B}_e\; G_1^{(e)'}/g_1'(0) \tag{3.48}$$

CHAPTER 4

RESULTS AND DISCUSSIONS

The equations for the first and second order boundary layer problems (3.30 - 31) and (3.34 - 36) have been integrated numerically by Runge-Kutta-Gill method on I.B.M. 7044 at Kanpur. The equations have been solved for various values of suction parameter C in the range of $-0.6 \leq C \leq 0.6$. The ranges for other parameters are $-7.5 \leq \Lambda_\ell$, Λ_t, $\Lambda_d \leq 4$, $\beta = 0$, 0.5, and 1 and $\sigma = 0.7$, 3 and 5. Results for second order contributions to skin friction $F''(0)$, heat transfer $G_1'(0)$ and recovery factor r_2 are displayed graphically. The results for $G_2'(0)$ can be obtained from those for $G_1'(0)$ and r_2.

For longitudinal curvature problem, solutions are shown in Figs. (2a-c) and (3a-c). Fig. (2a) shows skin friction $f''(0)$ verses Λ_ℓ for $C = 0.5$, and -0.5, $\beta = 0.5$ and $\sigma = 0.7$. For $\Lambda_\ell < 0$, we observe many singularities. The location of first singularity for $C = 0$ (without suction or injection) is also shown in the same graph by a vertical arrow (Afzal and Oberai 1972). The first critical value (approximately) of Λ_ℓ for $C = 0$ is at $\Lambda_\ell = -3.1$, for $C = 0.5$ (suction) at -3.5 and for $C = -.5$ (injection) at -2.72. The effect of suction on locations of singularities is to make the critical values of parameters Λ_ℓ more negative when compared to no suction case, i.e., the singularity is shifted towards left. For the case of injection the effect is otherwise. It may be noted that the effect of suction on locations of singularities is, qualitatively similar to those of favourable pressure gradients (Raisinghani, 1973). i.e., for a given suction if β increases, the

critical values move towards left. Figs. (2b) and (2c) show the effects of suction and injection on heat transfer $G_1'(0)$ and recovery factor r_2 respectively. Here the approximate location of first singularity for $C = 0$ is at $\Lambda_\ell = -2.75$, for $C = 0.5$ at $\Lambda_\ell = -3.0$ and for $C = -0.5$ at $\Lambda_\ell = -2.32$. It is to be observed that for a fixed value of suction parameter C, the singularity in $G_1'(0)$ and r_2 occurs at lower values of negative Λ_ℓ when compared tothe corresponding skin friction results. This is due to the fact that the eigen value for the homogeneous energy problem is lower than the corresponding momentum problem.

It has been pointed out by Raisinghani (1973) that the second order contributions for $\Lambda_\ell < 0$ are not of much practical importance and therefore detailed results have been computed for $\Lambda_\ell > 0$, and are shown in Figs. (3a,b,c). Fig. (3a) shows the effect of suction and injection on second order skin friction and displacement thickness. It is seen that for $\Lambda_\ell = 0$ and for a given β, suction increases (in magnitude) the skin friction while injection decreases (in magnitude) it. Fig. (3b) shows that for $\sigma = 0.7$, the second order heat transfer $G_1'(0)$ decreases with suction and increases with injection. However for high Prandtle numbers ($\sigma = 3, 5$), the trend of heat transfer curve is reversed after a certain value of injection i.e., it starts decreasing. The rate of decrease becomes more pronounced as σ increases. Fig. (3c) shows results for recovery factor r_2. As C increases, the magnitude of recovery factor decreases monotonically.

For transverse curvature problem, the results are shown in Figs. (4a,b,c) and Figs. (5a,b,c). Figs. (4a,b,c) show the effects of suction and injection on singularities in the solutions of second order

skin friction, heat transfer and recovery factor respectively. Here the said effects are similar to those discussed in the case of longitudinal curvature problem and hence need no additional comments. Fig. (5a) shows results for second order skin friction and displacement thickness. For $\beta = 0$, suction reduces the skin friction while injection increases it. For $\beta = .5$ and 1, skin friction decreases with suction (at a lower rate as compared to that for $\beta = 0$) while with injection it increases, reaches a maximum and then decreases. The nature of the second order heat transfer curves, as shown in Fig. (5b), is similar to those for longitudinal curvature problem. Fig. (5c) shows the effect of suction and injection on second order recovery factor. For $\beta = 0$, $\sigma = 3$ and 5, the magnitude of r_2 first increases, reaches a maximum and then decreases very rapidly as C decreases.

For displacement speed problem, the results are shown in Figs. (6a,b,c) and (7a,b,c). Figs. (6a,b,c) show the effect of suction and injection on locations of singularities for skin friction, heat transfer and recovery factor respectively. The behaviour of the singularities is similar to those discussed earlier in the case of longitudinal and transverse curvature problems and hence no further comments are needed.

Fig. (7a) shows variation of second order skin friction and displacement thickness verses suction parameter C. Skin friction increases monotonically with C. For a given value of C, $\sigma (= .7)$ and $\beta (= .5)$, the skin friction increases as Λ_d increases. The magnitude of second order heat transfer, as shown in Fig. (7b), decreases with increasing values of suction and injection. For a fixed value of C,

β (= .5), σ (= .7), the heat transfer increases with Λ_d. Fig. (7c) shows results for second order recovery factor. The recovery factor r_2 increases monotonically as C decreases. For $\beta = 0$, $\sigma = 3$ and 5, r_2 increases, reaches a maximum and then decreases very rapidly as C decreases. For a given C, σ (= .7) and β(= .5), the recovery factor increases with Λ_d.

For external vorticity problem, the results are shown in Figs. (8a,b,c). The second order skin friction, as shown in Fig. (8a) decreases monotonically with suction parameter C. For a given value of C, the skin friction increases as β decreases. Fig. (8b) shows results for second order heat transfer. For σ = .7, the magnitude of heat transfer decreases monotonically with C. For σ = 3 and 5, the magnitude of heat transfer increases, reaches a maximum and then decreases. The second order recovery factor r_2 shown in Fig.(8c) decreases monotonically as suction parameter C increases. For $\beta = 0$, $\sigma = 5$ the curve shows a maxima and decreases as injection (-C) increases.

For temperature gradient problem, the results are shown in Figs. (9a) and (9b). The second order heat transfer increases monotonically with parameter C. For a given value of C and β, the recovery factor increases with σ. From Fig. (9b), it is observed that the magnitude of second order recovery factor increases monotonically with C. Further for a given value of C, β, the magnitude of recovery factor increases as σ decreases.

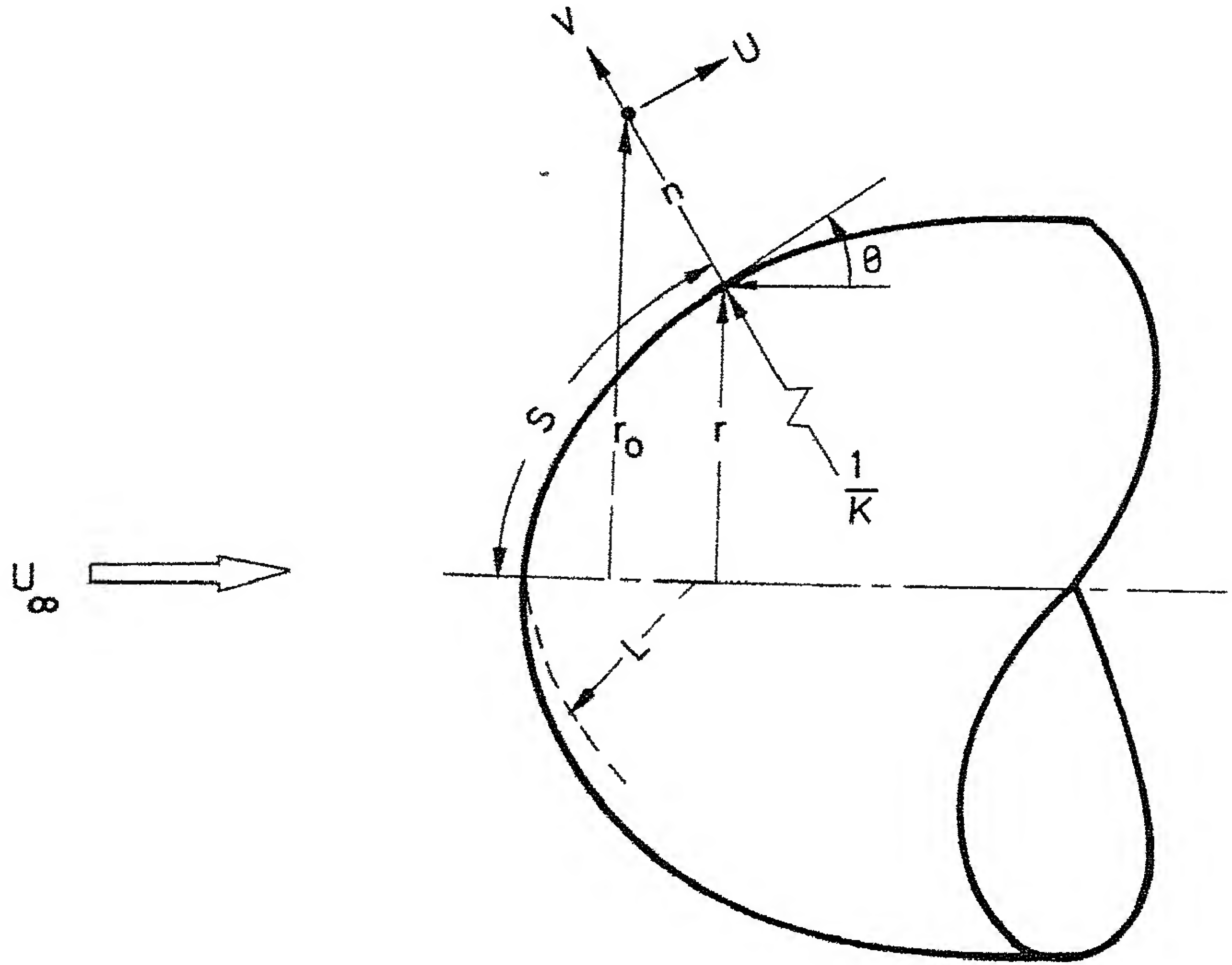

FIG 1 COORDINATE SYSTEM

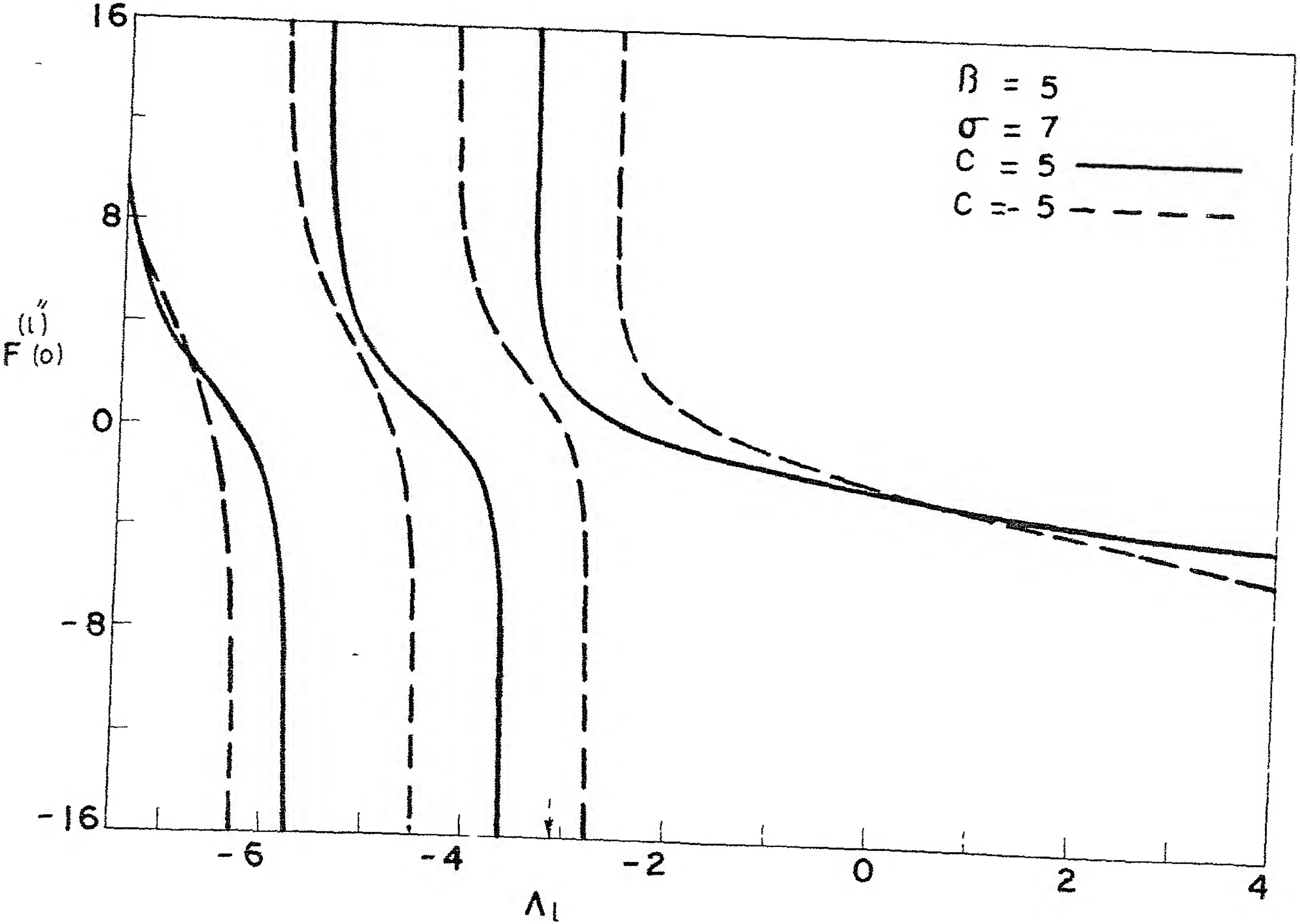

FIG 2a LONGITUDINAL CURVATURE SOLUTIONS. EFFECTS OF SUCTION AND INJECTION ON LOCATIONS OF SINGULARITIES IN SKIN FRICTION

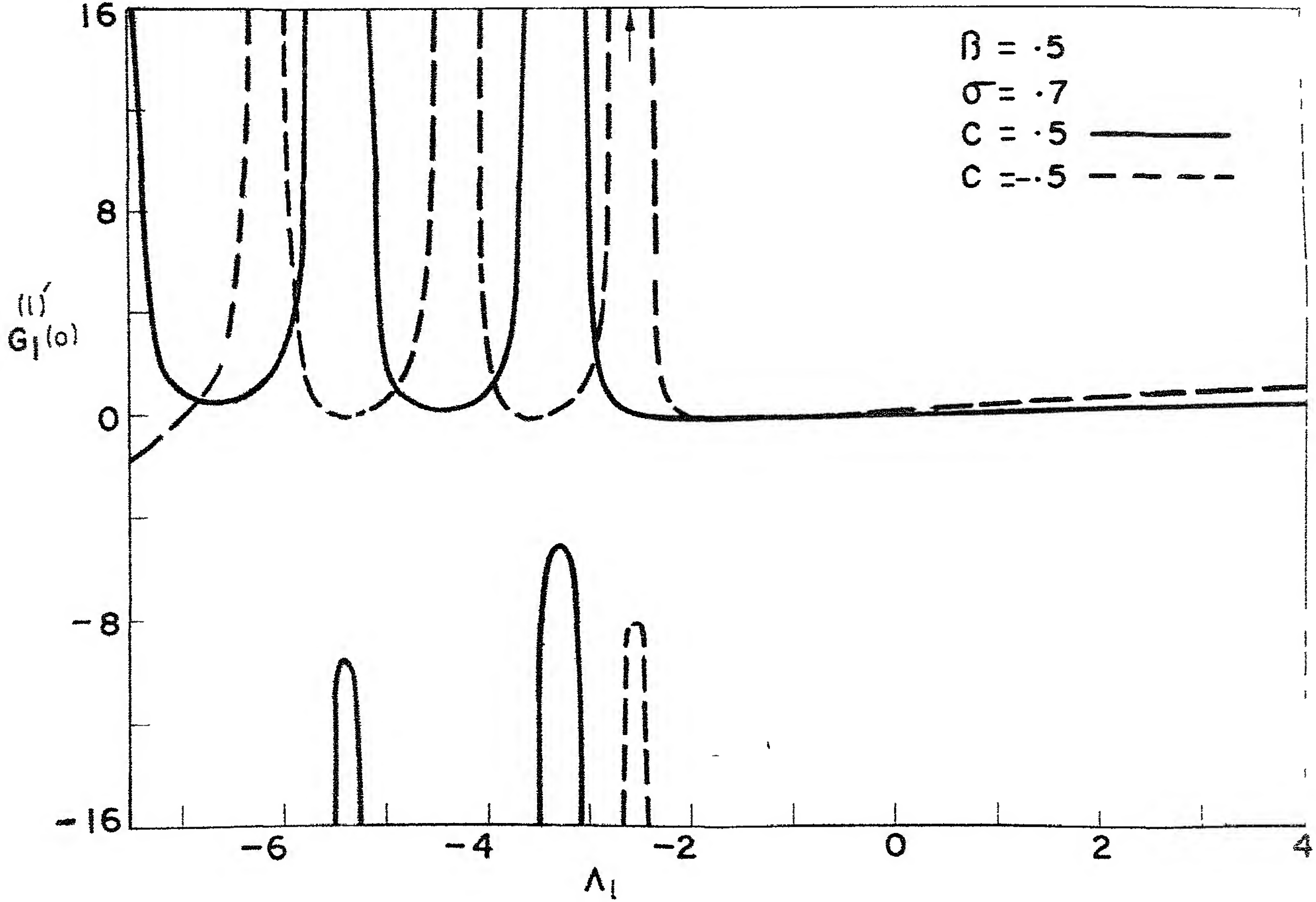

FIG 2b LONGITUDINAL CURVATURE SOLUTIONS EFFECTS OF SUCTION AND [illegible] ON LOCATIONS OF SINGULARITIES IN HEAT TRANSFER

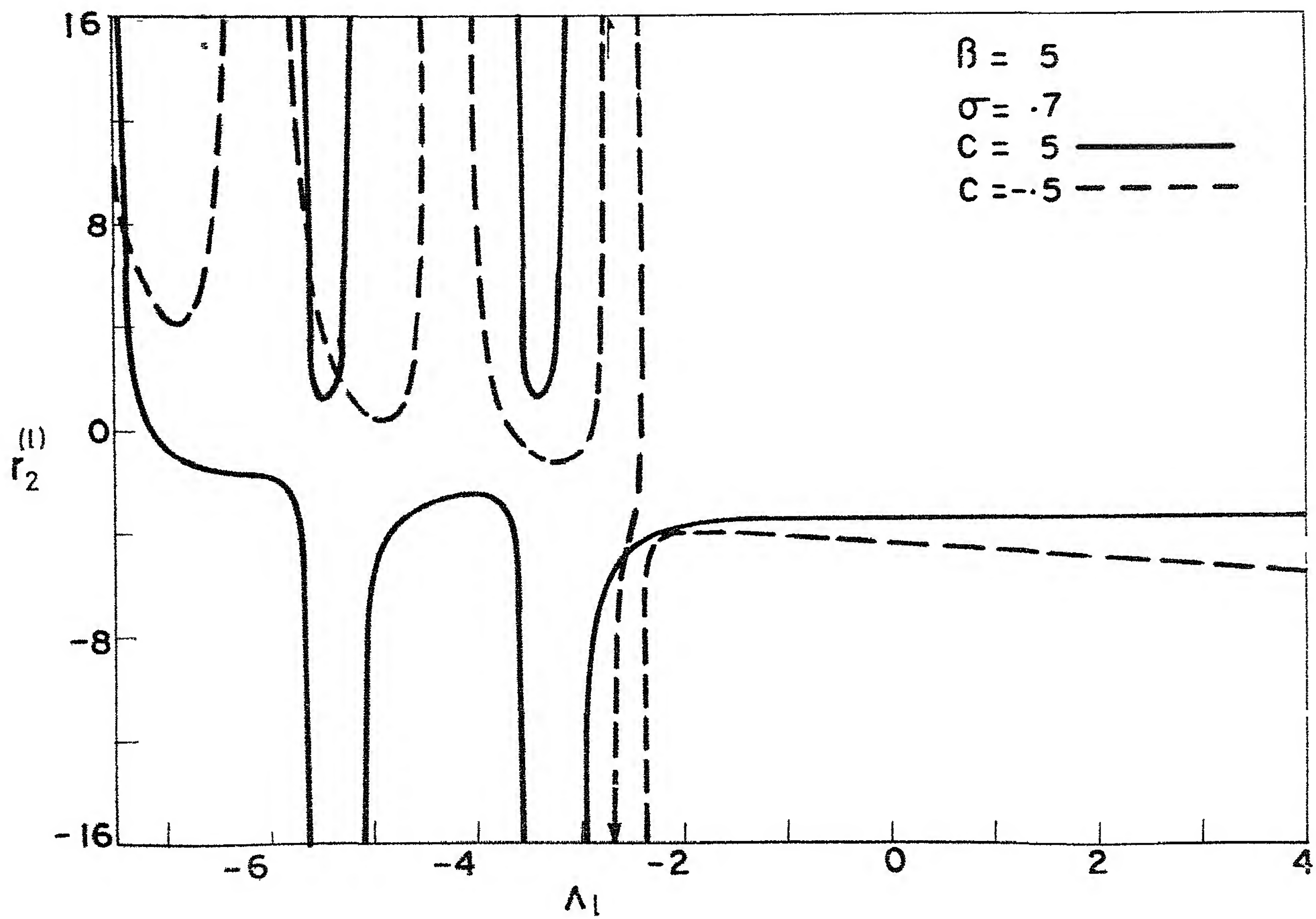

FIG 2c LONGITUDINAL CURVATURE SOLUTIONS EFFECTS OF SUCTION AND INJECTION ON LOCATIONS OF SINGULARITIES ON RECOVERY FACTOR

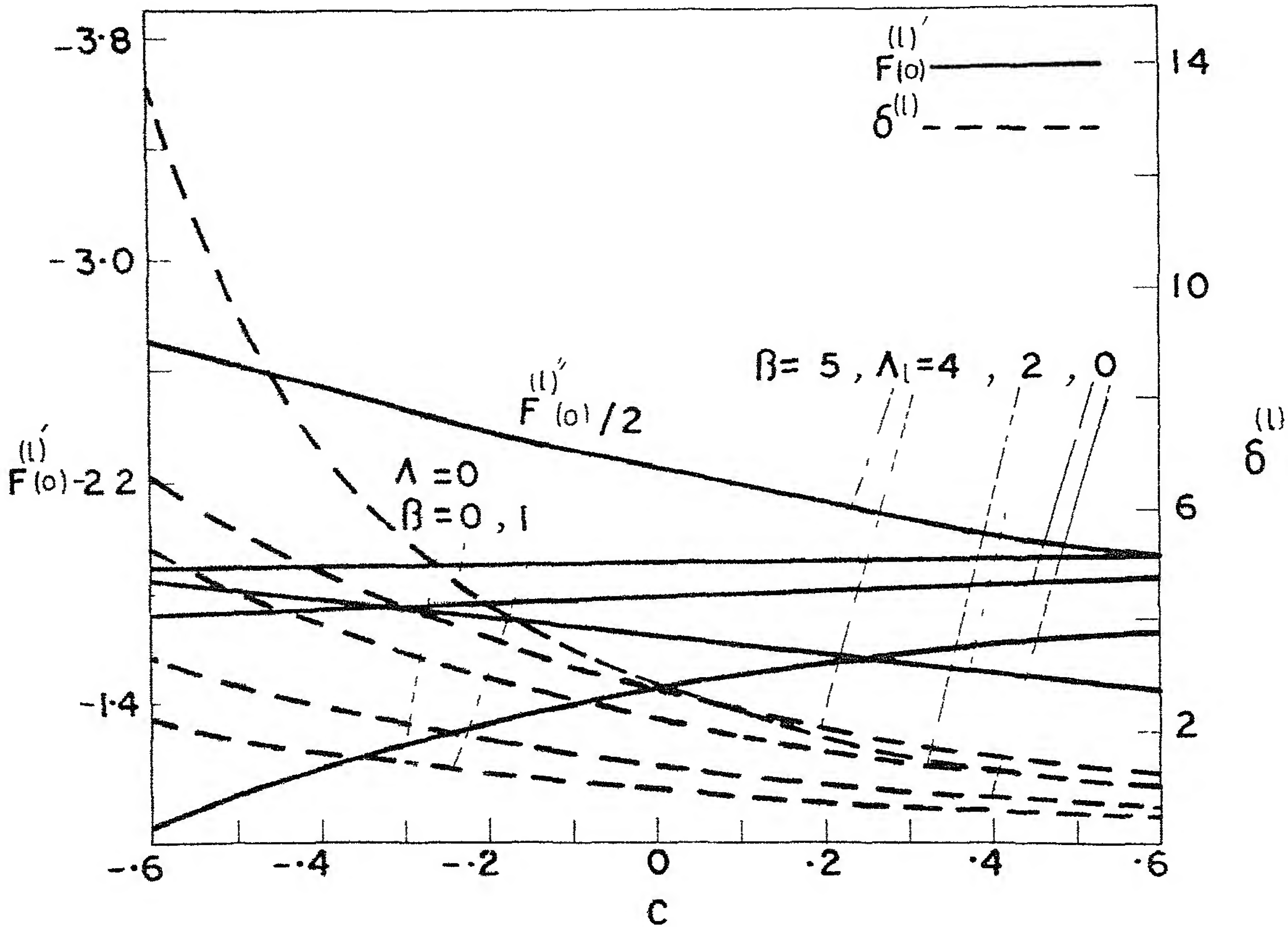

FIG 3a LONGITUDINAL CURVATURE SOLUTIONS: EFFECTS OF SUCTION AND INJECTION ON SKIN FRICTION AND DISPLACEMENT THICKNESS

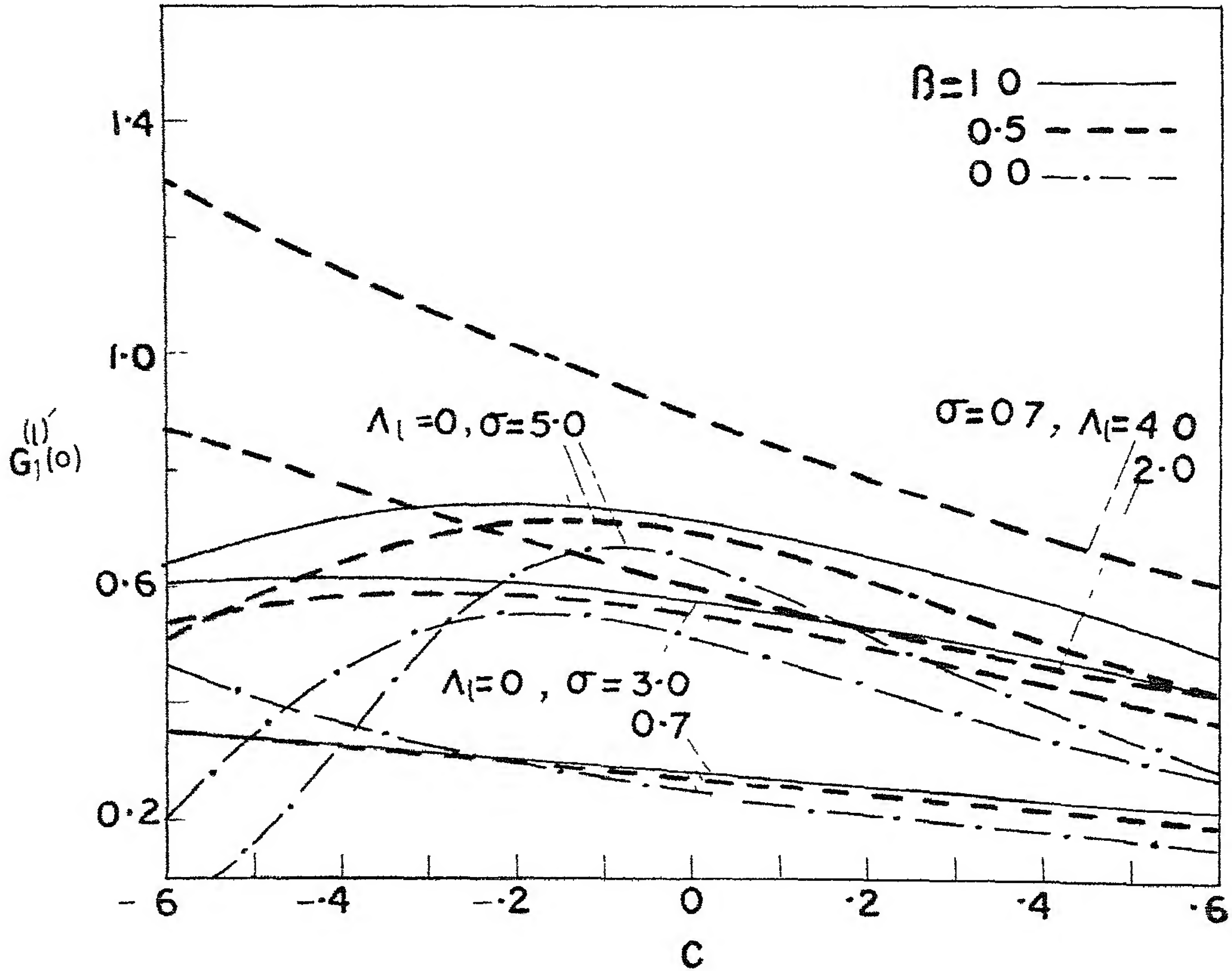

FIG 3b LONGITUDINAL CURVATURE SOLUTIONS EFFECTS OF SUCTION AND INJECTION ON HEAT TRANSFER

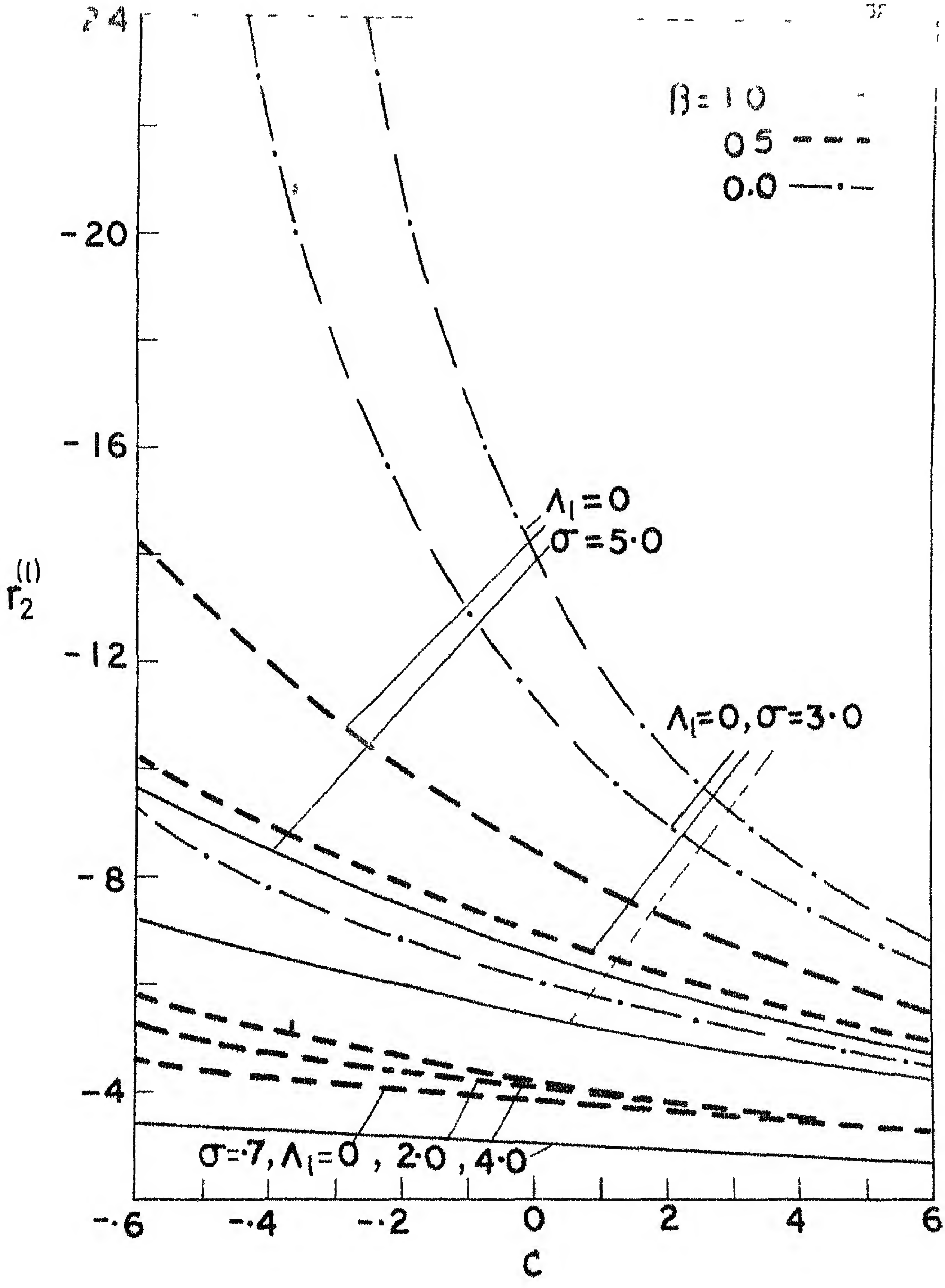

FIG 3c LONGITUDINAL CURVATURE SOLUTIONS EFFECTS OF SUCTION AND INJECTION ON RECOVERY FACTOR

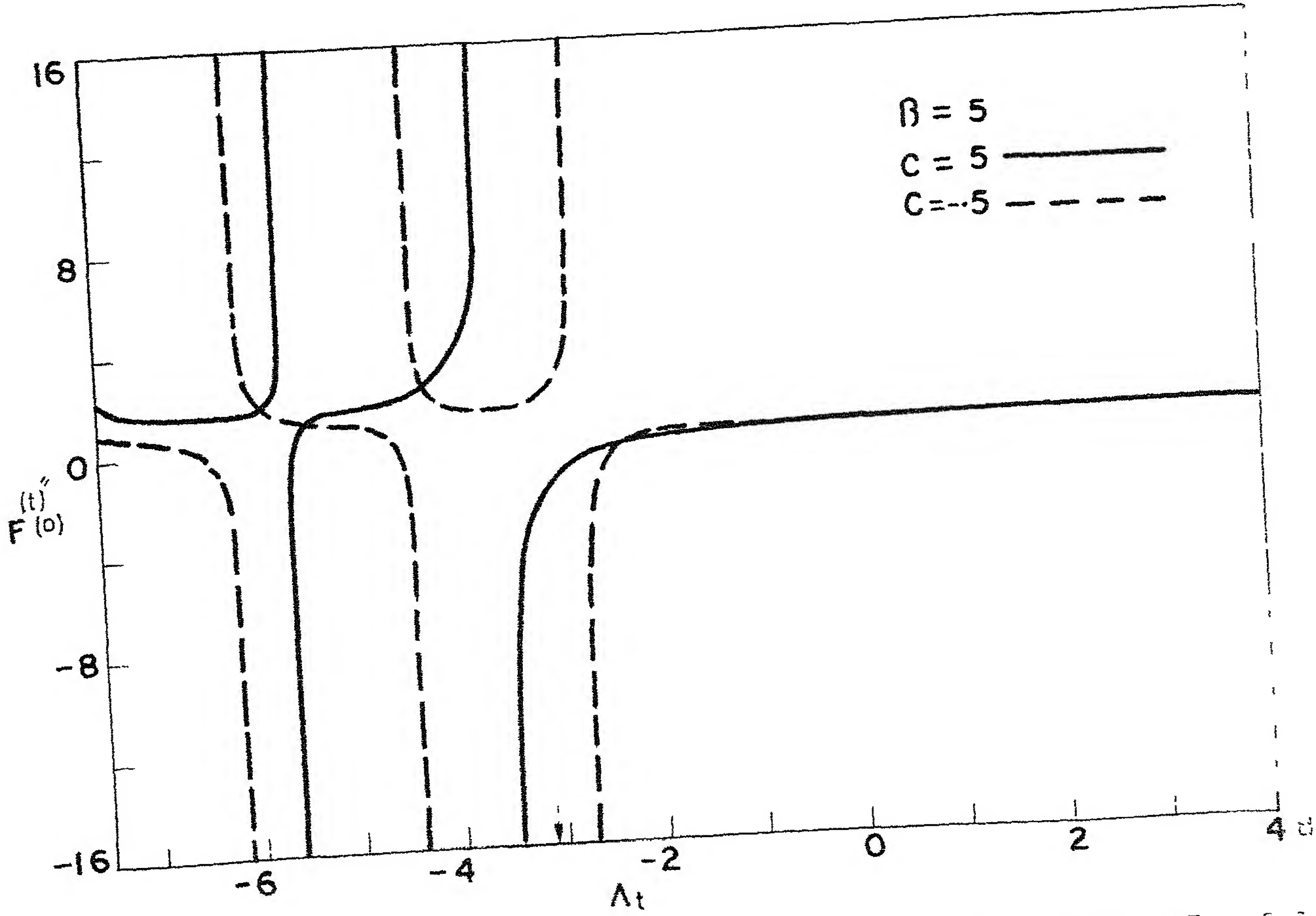

FIG 4a TRANS.E OF CUF ATURE SOL..TONS: EFFECTS OF SUCTION AND ... S OF S N ARIES IN SKIN FRICTION

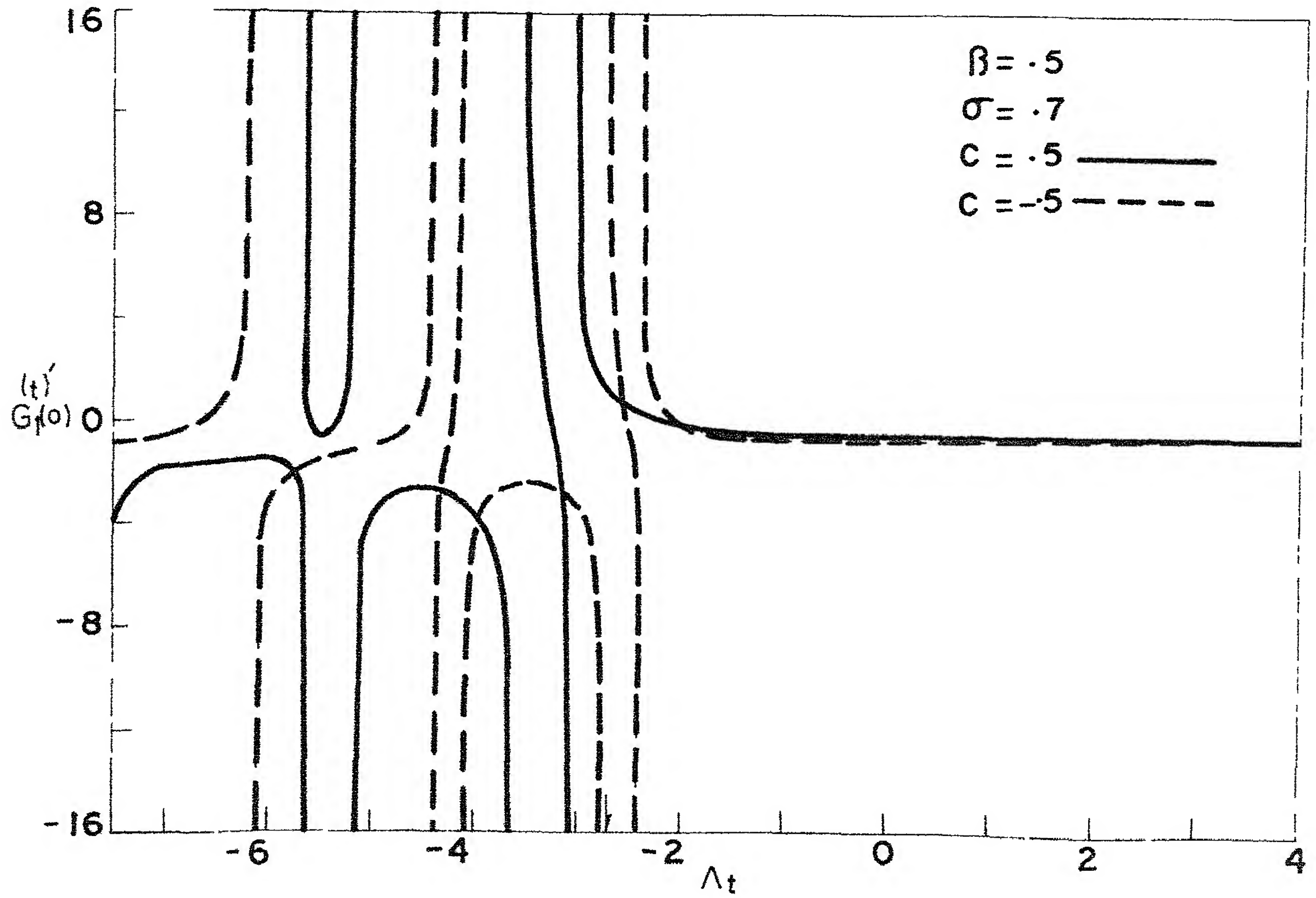

FIG 4b TRANSVERSE CURVATURE SOLUTIONS: EFFECTS OF SUCTION AND INJECTION ON LOCATIONS OF SINGULARITIES IN HEAT TRANSFER

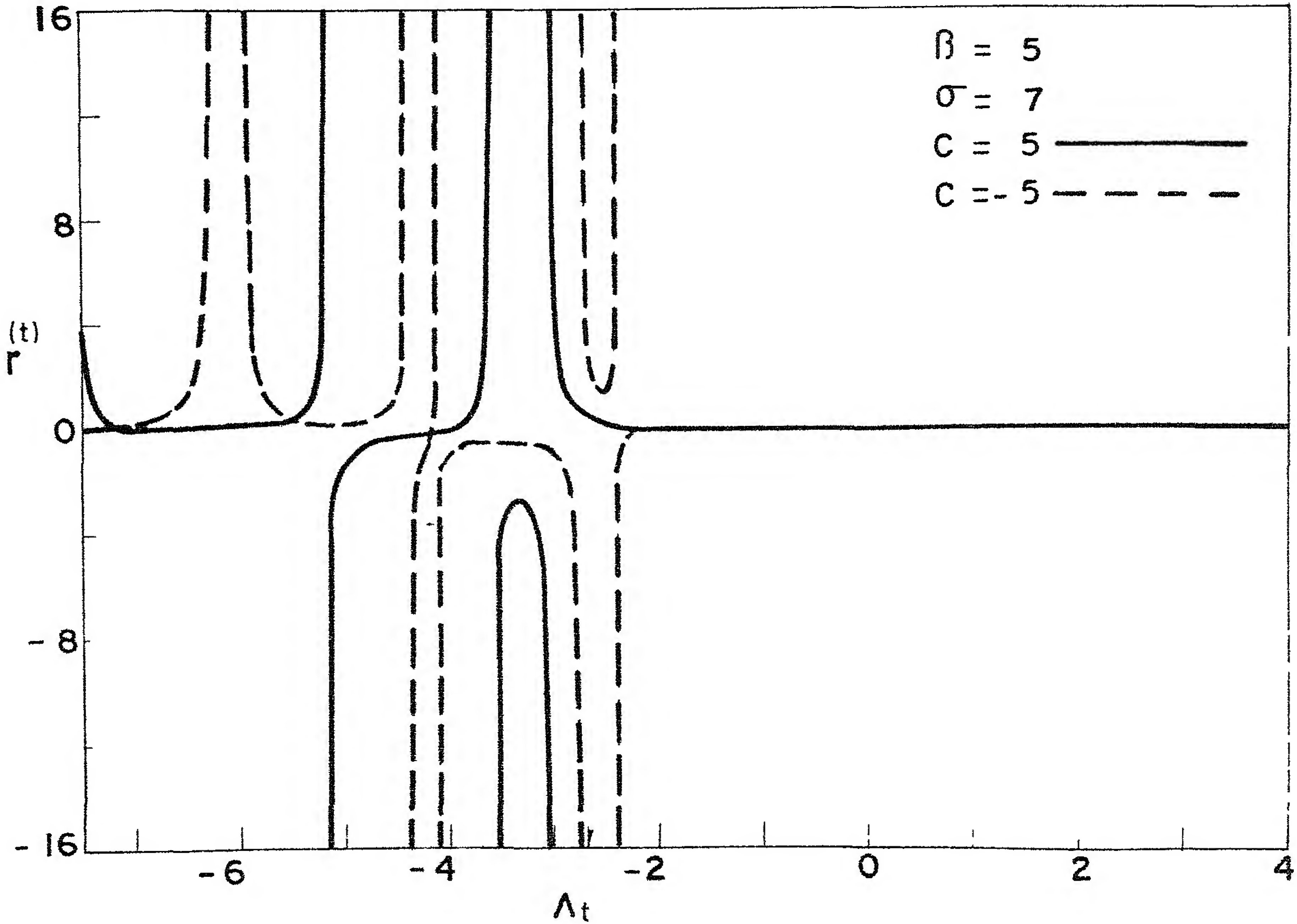

FIG 4c TRANSVERSE CURVATURE SOLUTIONS. EFFECTS OF SUCTION AND INJECTION ON LOCATIONS OF SINGULARITIES IN RECOVERY FACTOR

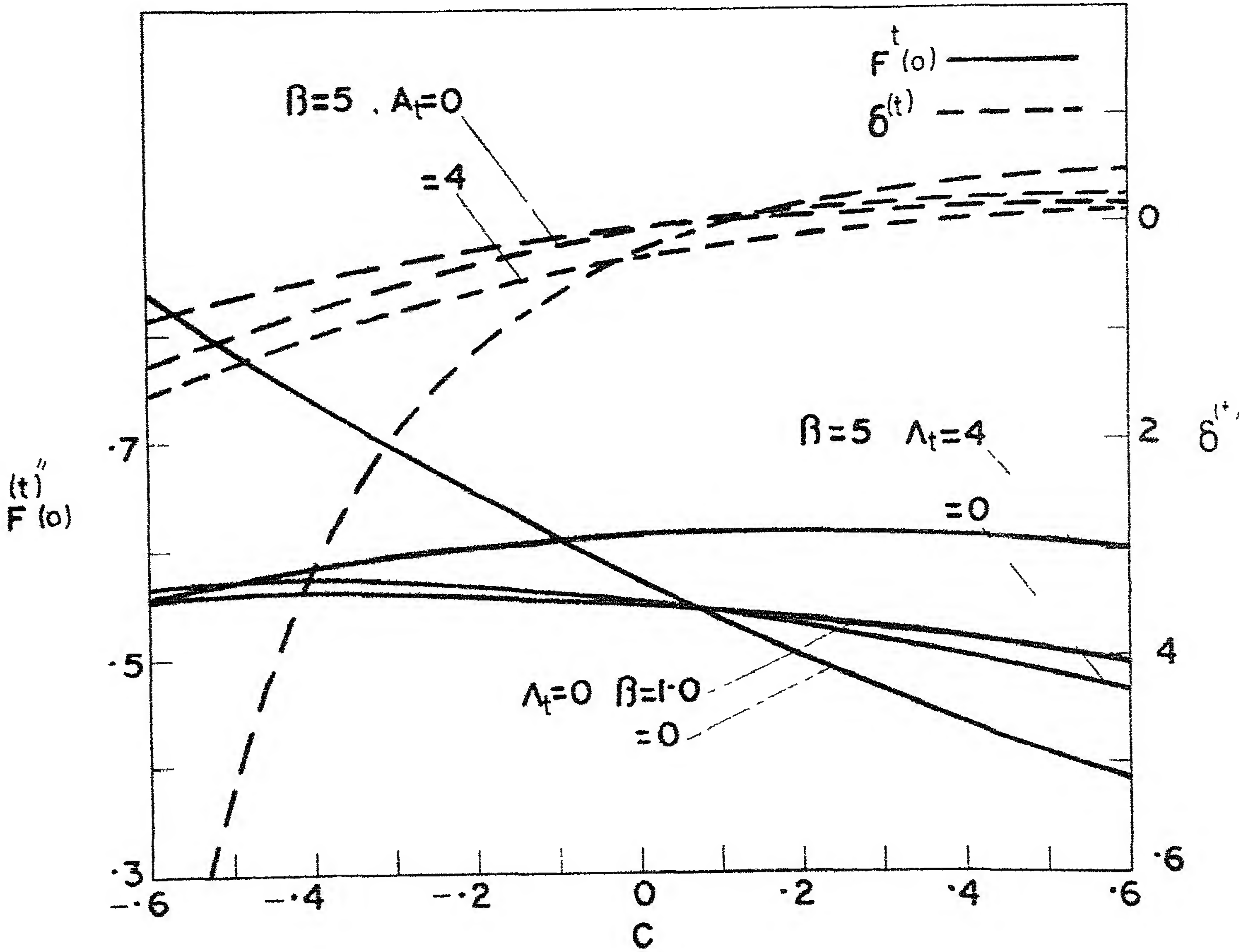

FIG 5a TRANSVERSE CURVATURE SOLUTIONS. EFFECTS OF SUCTION AND INJECTION ON SKIN FRICTION AND DISPLACEMENT THICKNESS

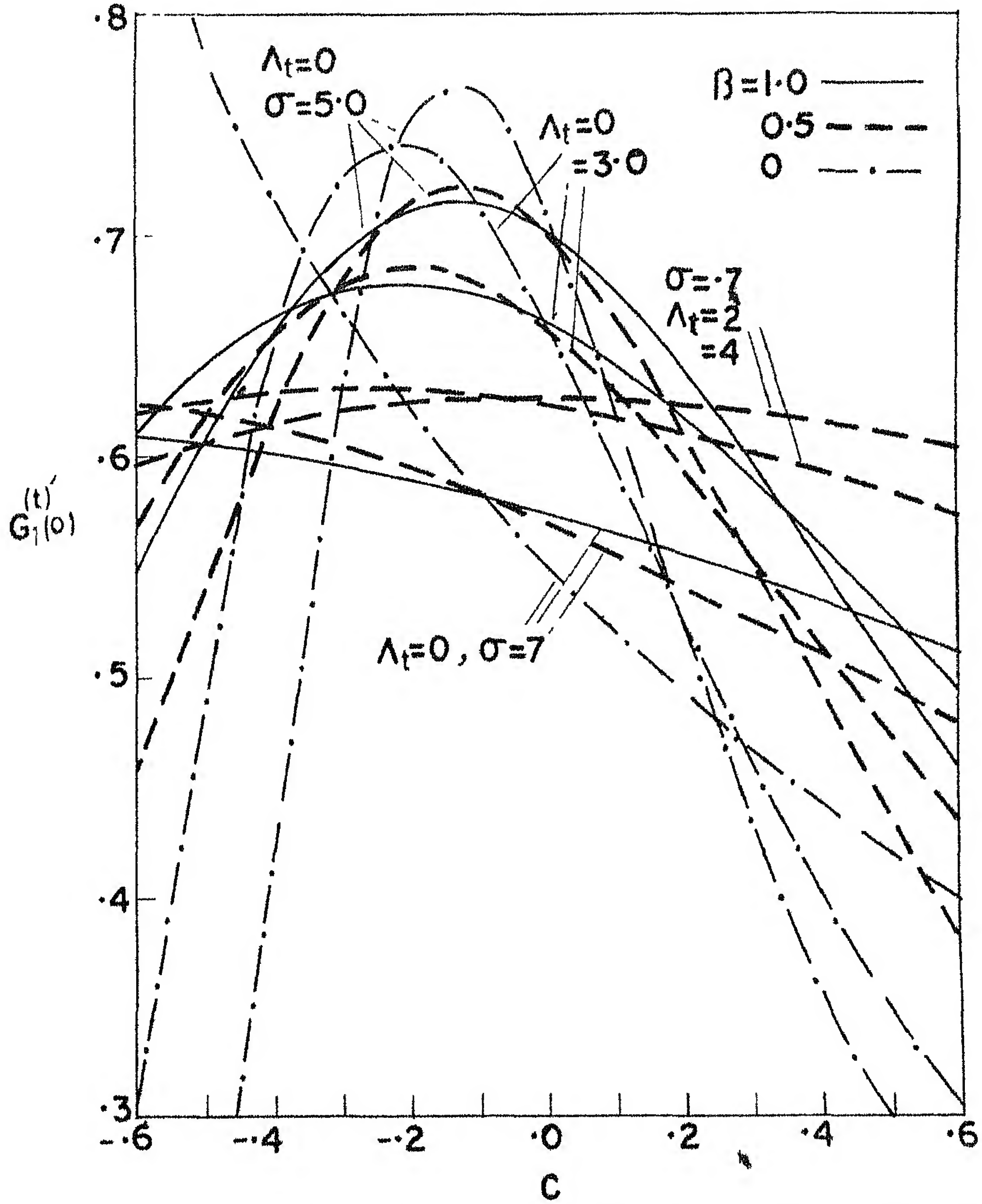

FIG 5b TRANSVERSE CURVATURE SOLUTIONS EFFECTS OF SUCTION AND INJECTION ON HEAT TRANSFER

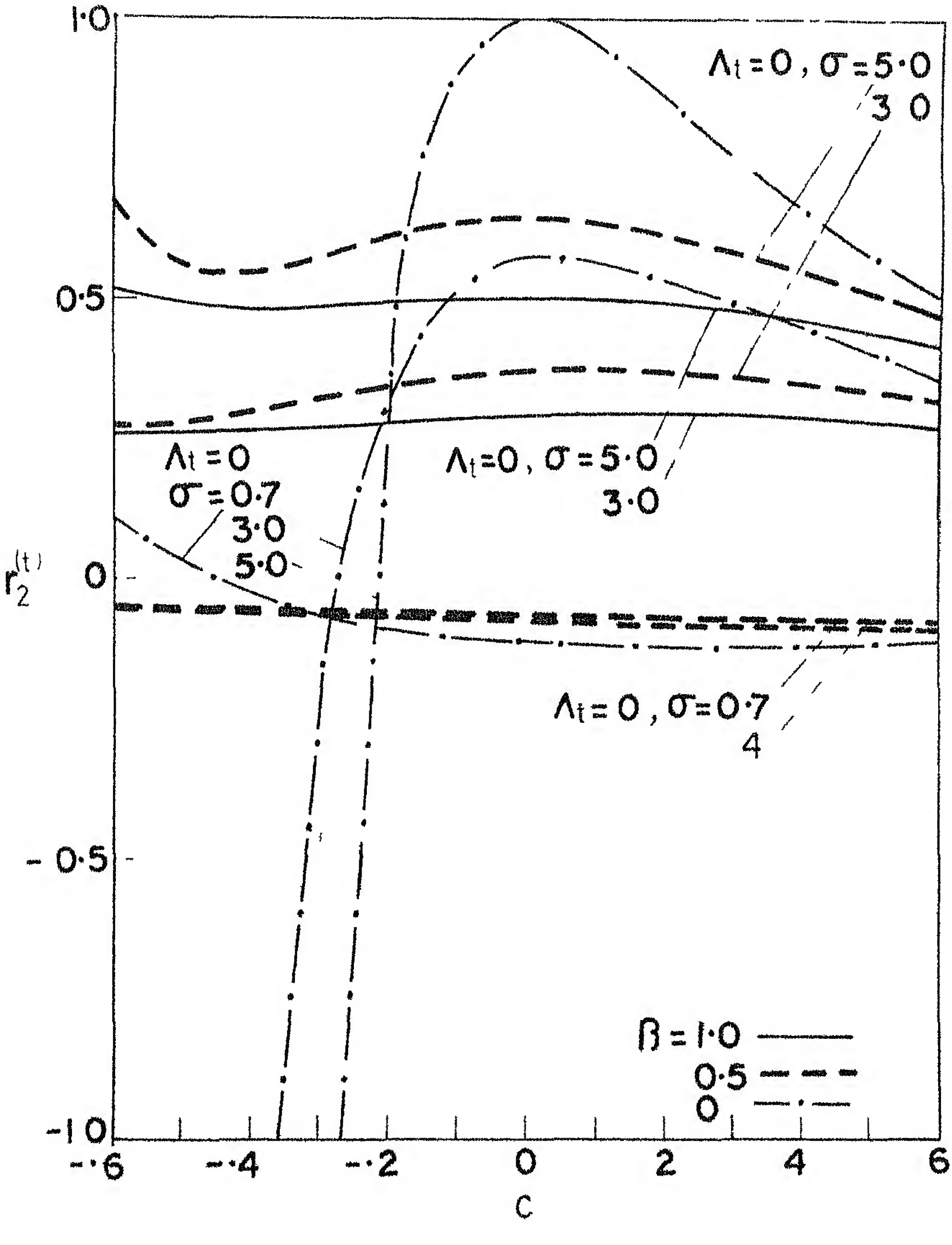

FIG 5 c TRANSVERSE CURVATURE SOLUTIONS: EFFECTS OF SUCTION AND INJECTION ON RECOVERY FACTOR

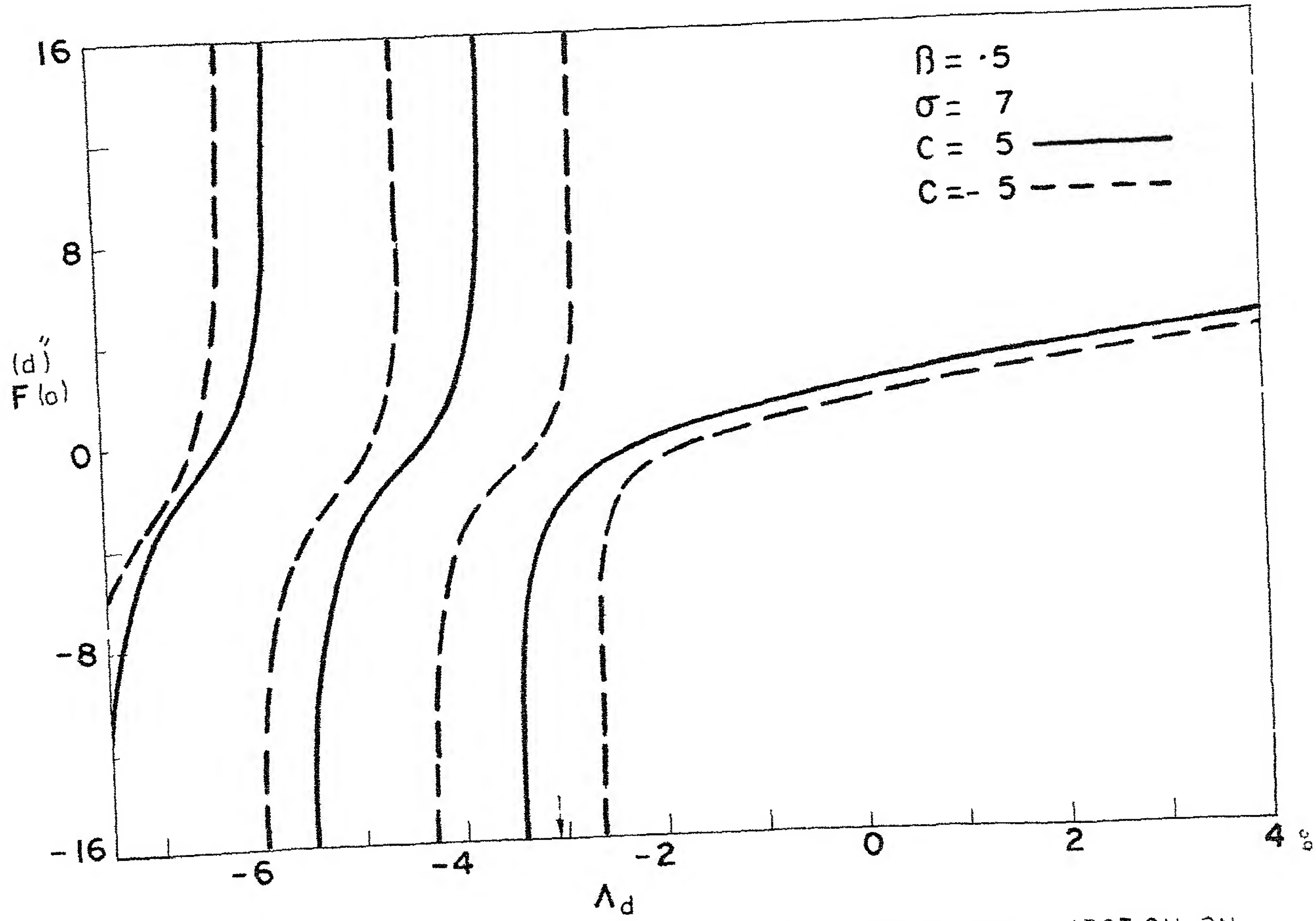

FIG 6a DISPLACEMENT SPEED SOLUTIONS. EFFECTS OF SUCTION AND INJECTION ON LOCATIONS OF SINGULARITIES IN SKIN FRICTION

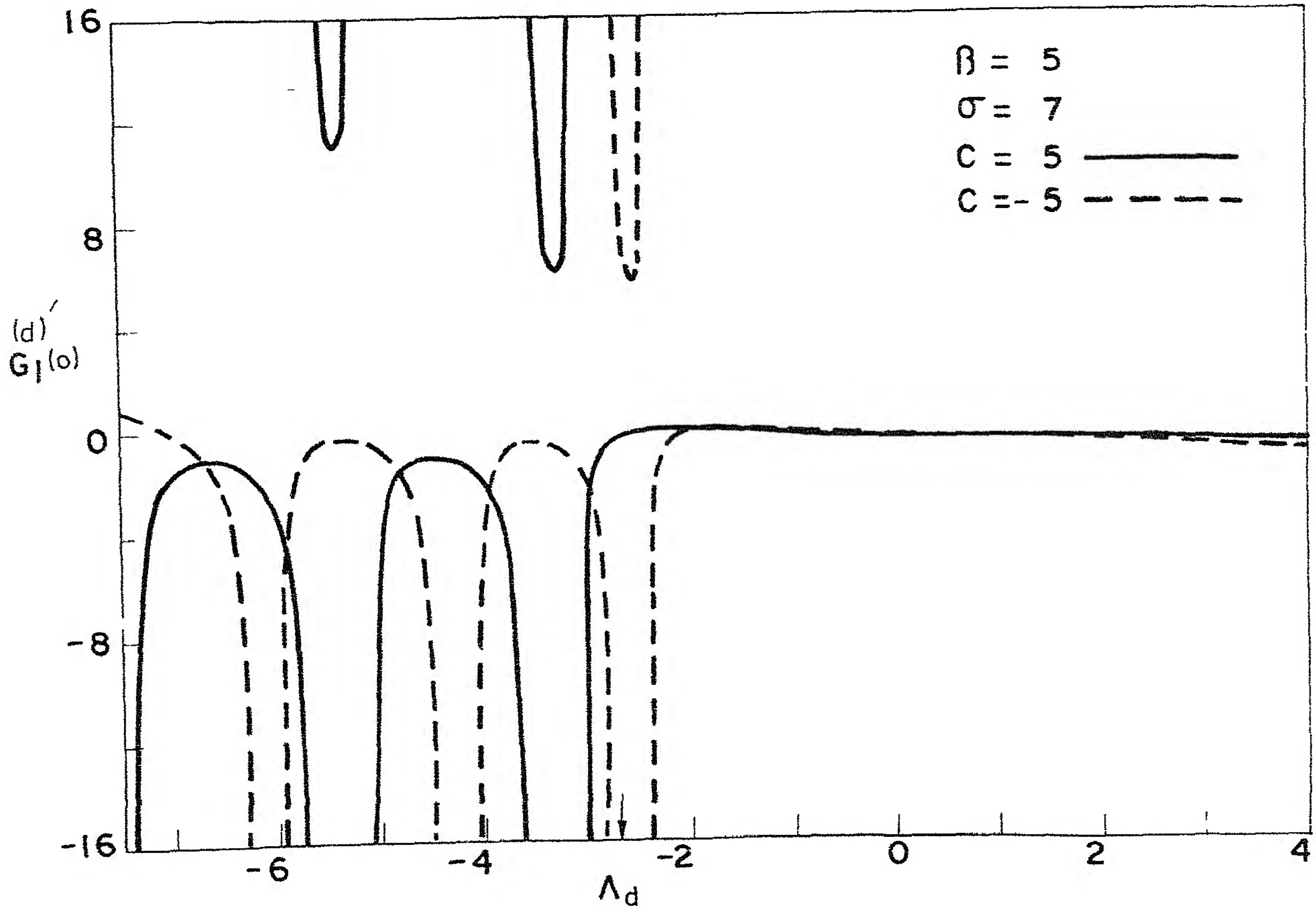

FIG 6b DISPLACEMENT SPEED SOLUTIONS: EFFECTS OF SUCTION AND INJECTION ON LOCATIONS OF SINGULARITIES IN HEAT TRANSFER

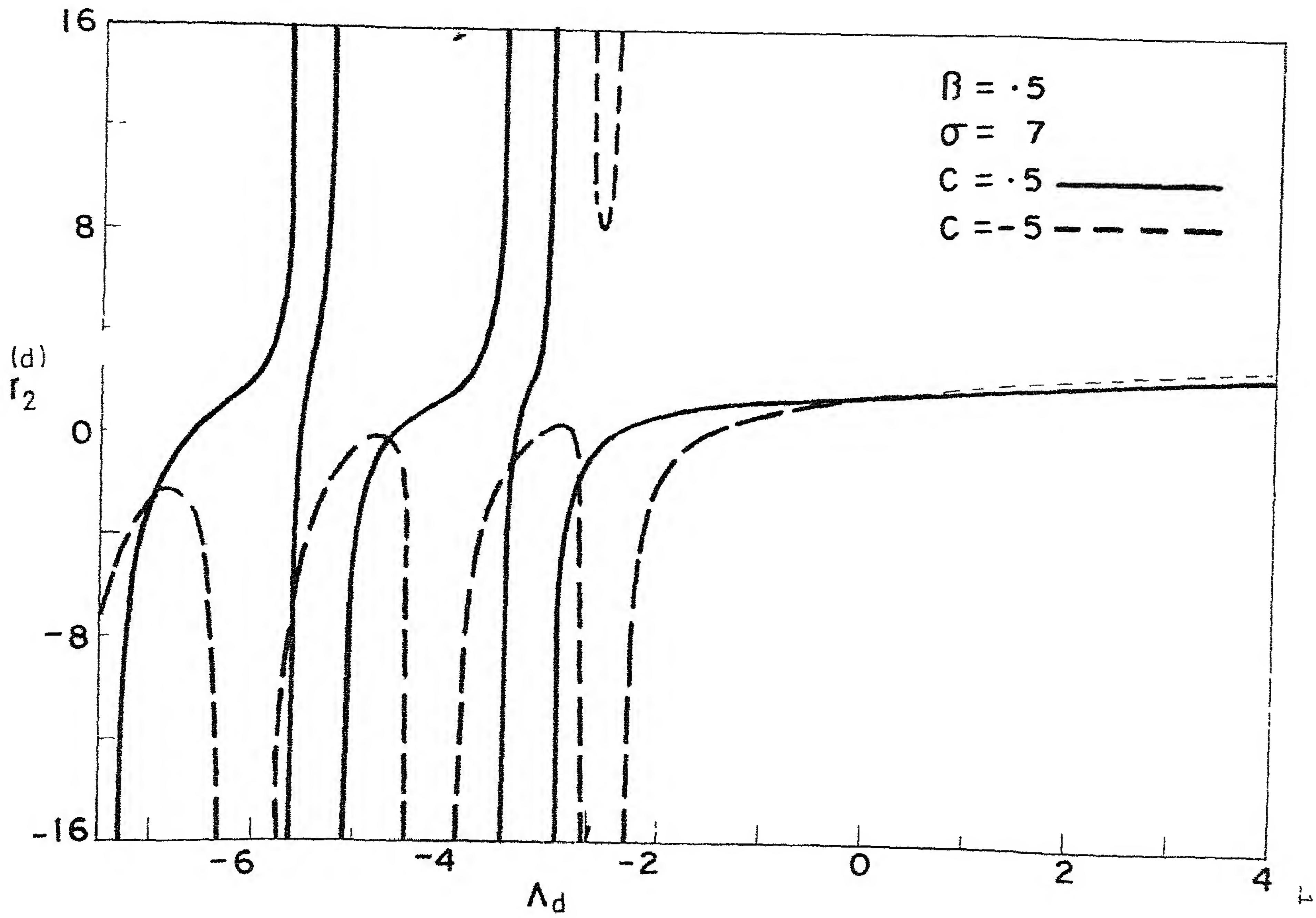

FIG 6c DISPLACEMENT SPEED SOLUTIONS. EFFECTS OF SUCTION AND INJECTION ON LOCATIONS OF SINGULARITIES IN RECOVERY FACTOR

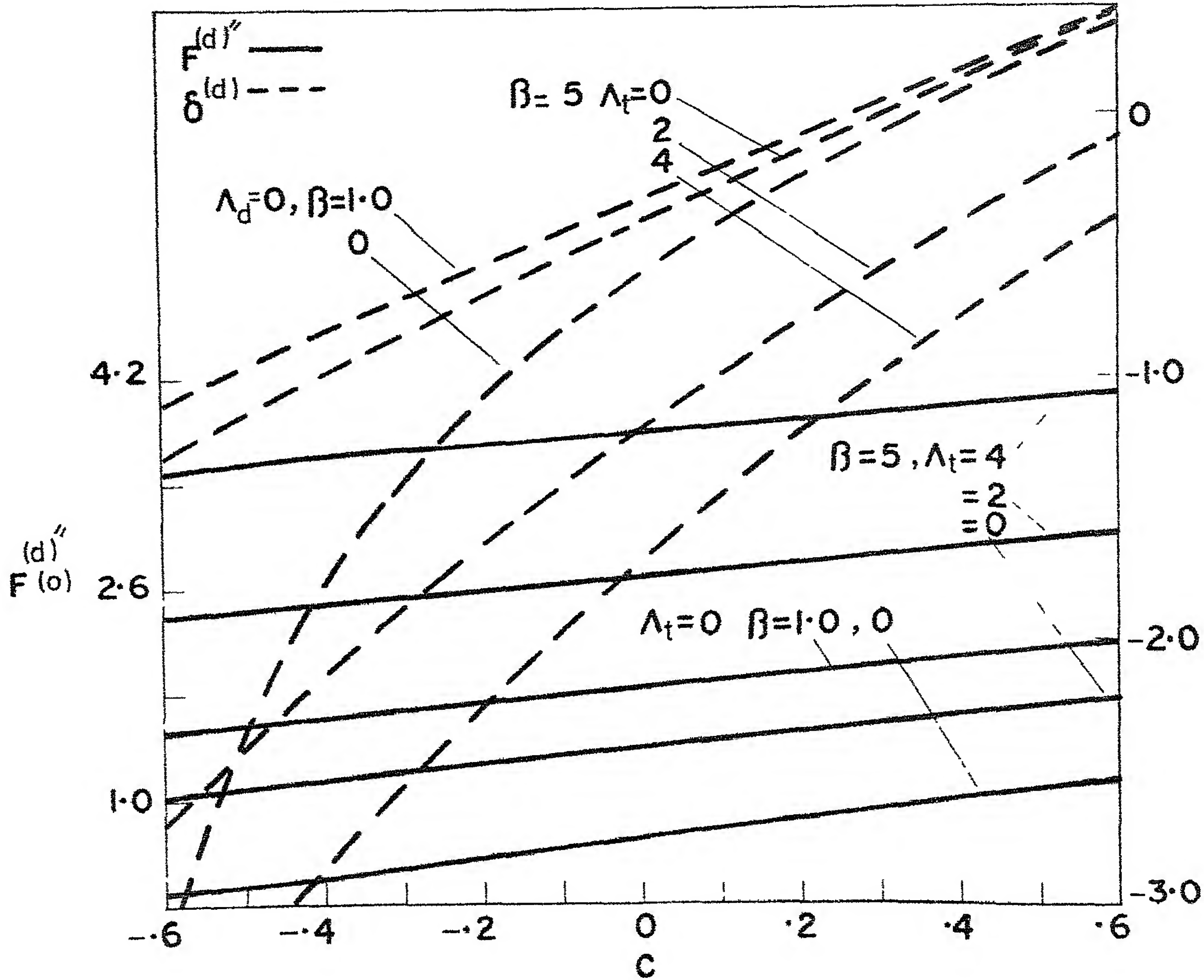

FIG 7c DISPLACEMENT SPEED SOLUTIONS EFFECTS OF SUCTION AND EFFECT ... ON SKIN FRICTION ... DISPLACEMENT THICKNESS

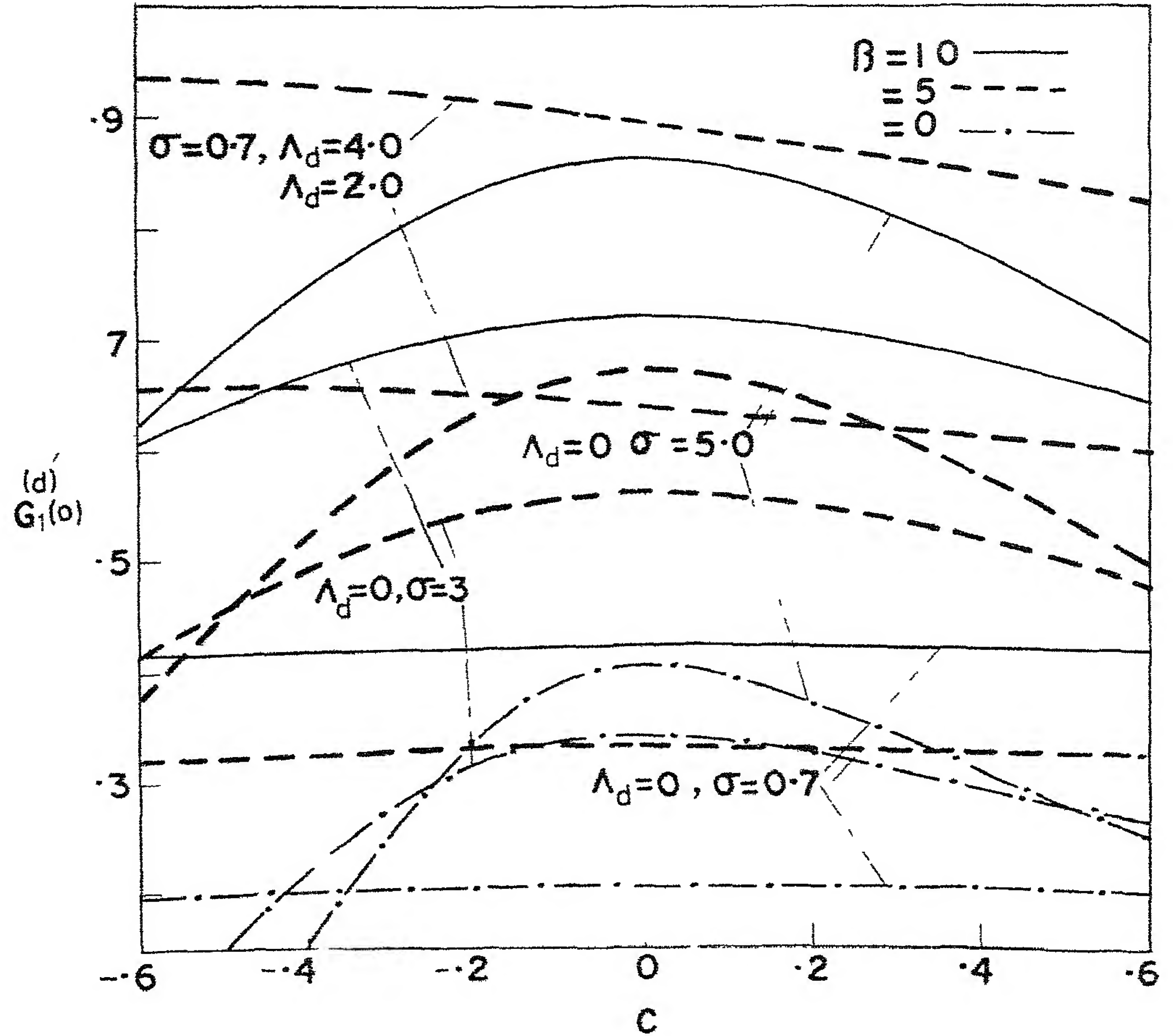

FIG 7b DISPLACEMENT SPEED SOLUTIONS: EFFECTS OF SUCTION

... TRANSFER

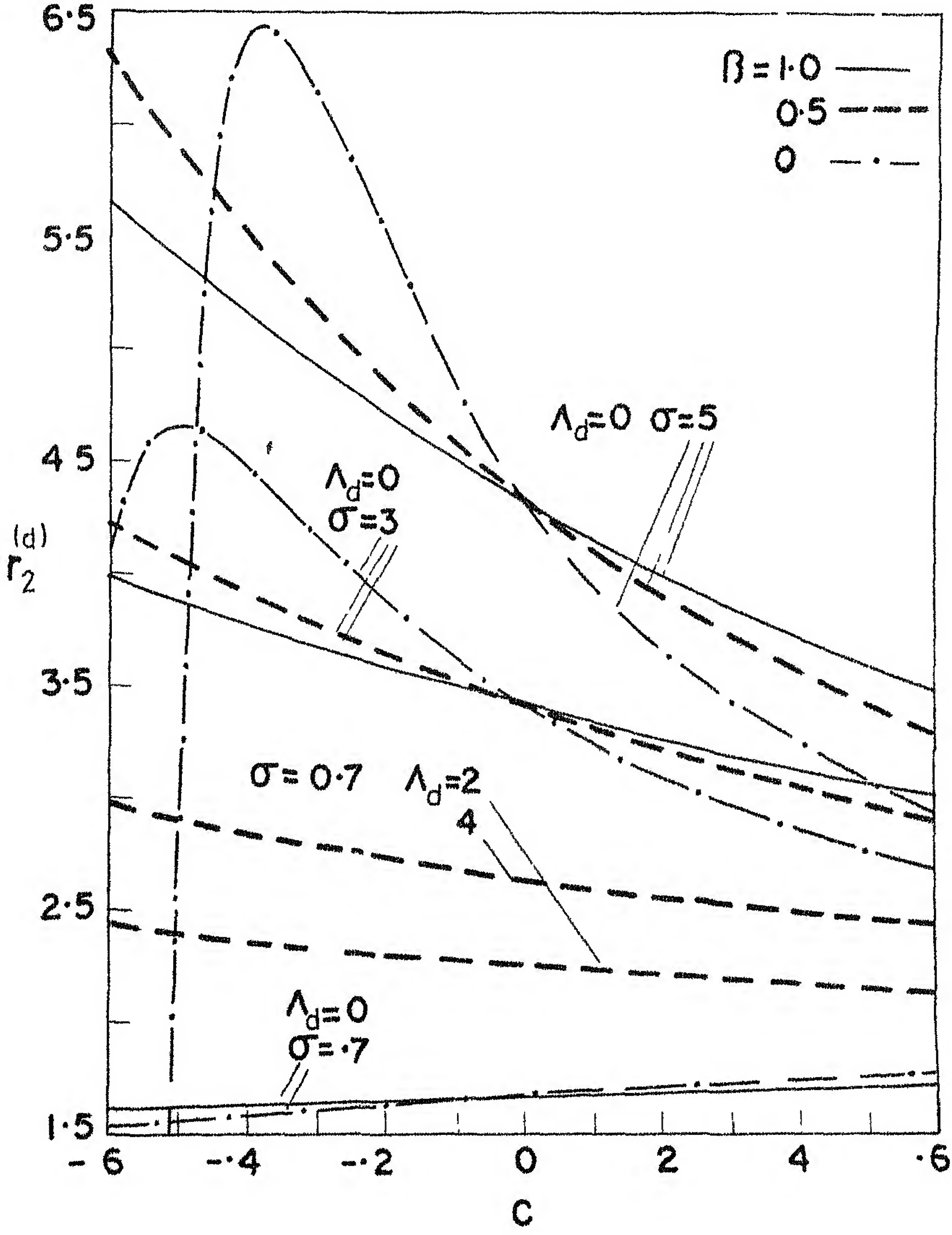

FIG 7C DISPLACEMENT SPEED SOLUTIONS EFFECTS OF SUCTION AND INJECTION ON RECOVERY FACTOR

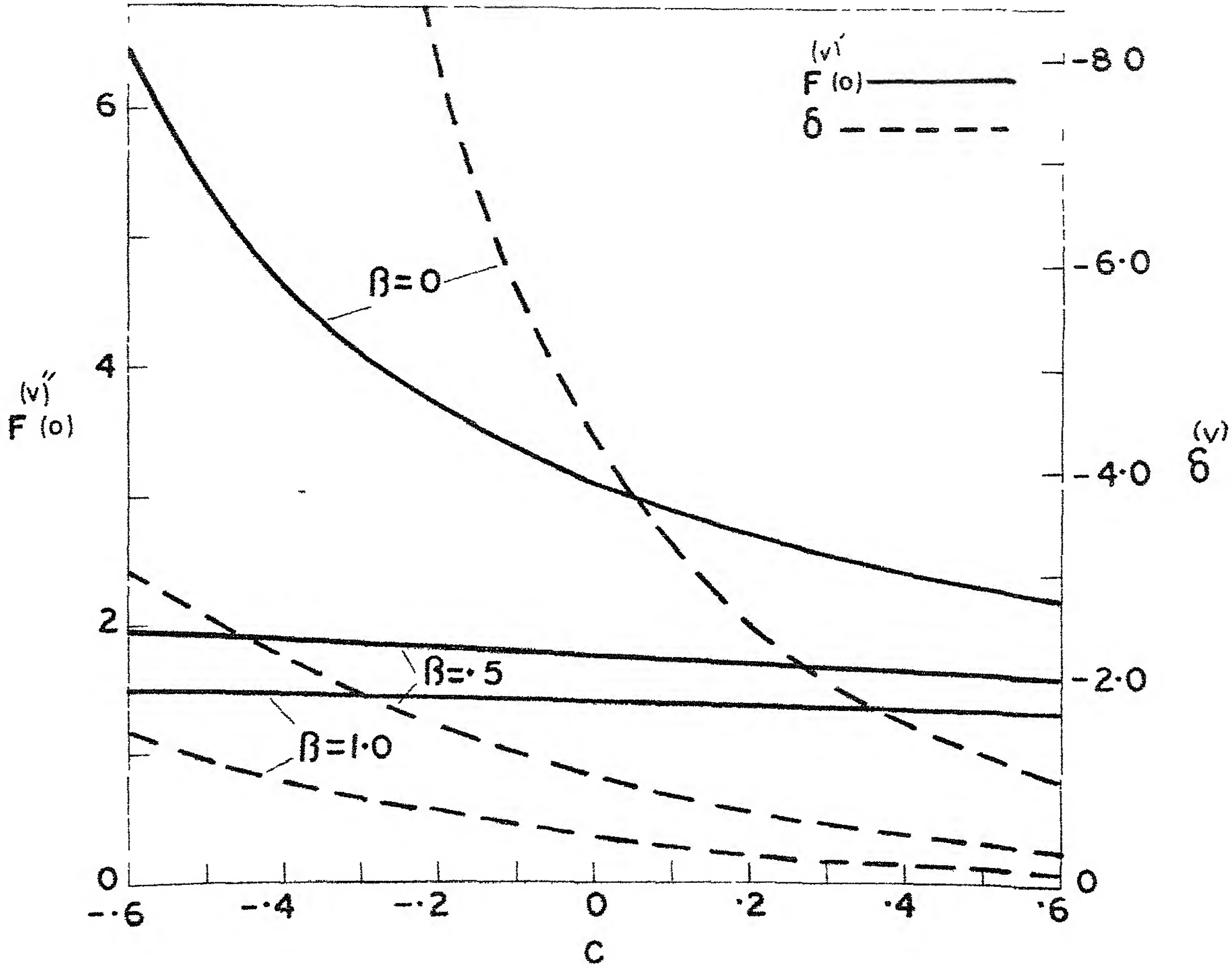

FIG 5a EXTERNAL VORTICITY SOLUTIONS EFFECTS OF SUCTION AND [illegible] AND DISPLACEMENT THICKNESS

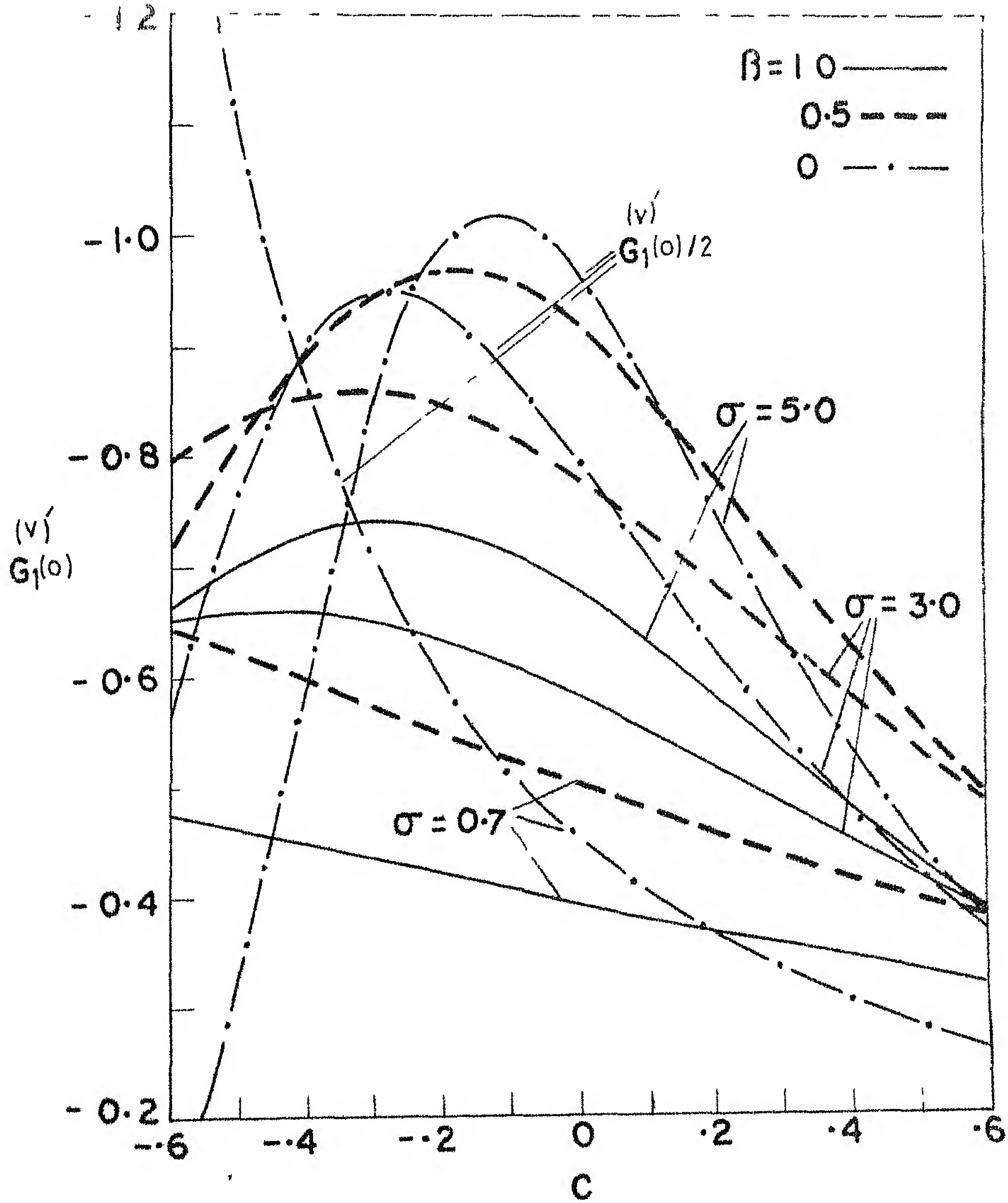

FIG. 8 b EXTERNAL VORTICITY SOLUTIONS EFFECTS OF SUCTION AND INJECTION ON HEAT TRANSFER

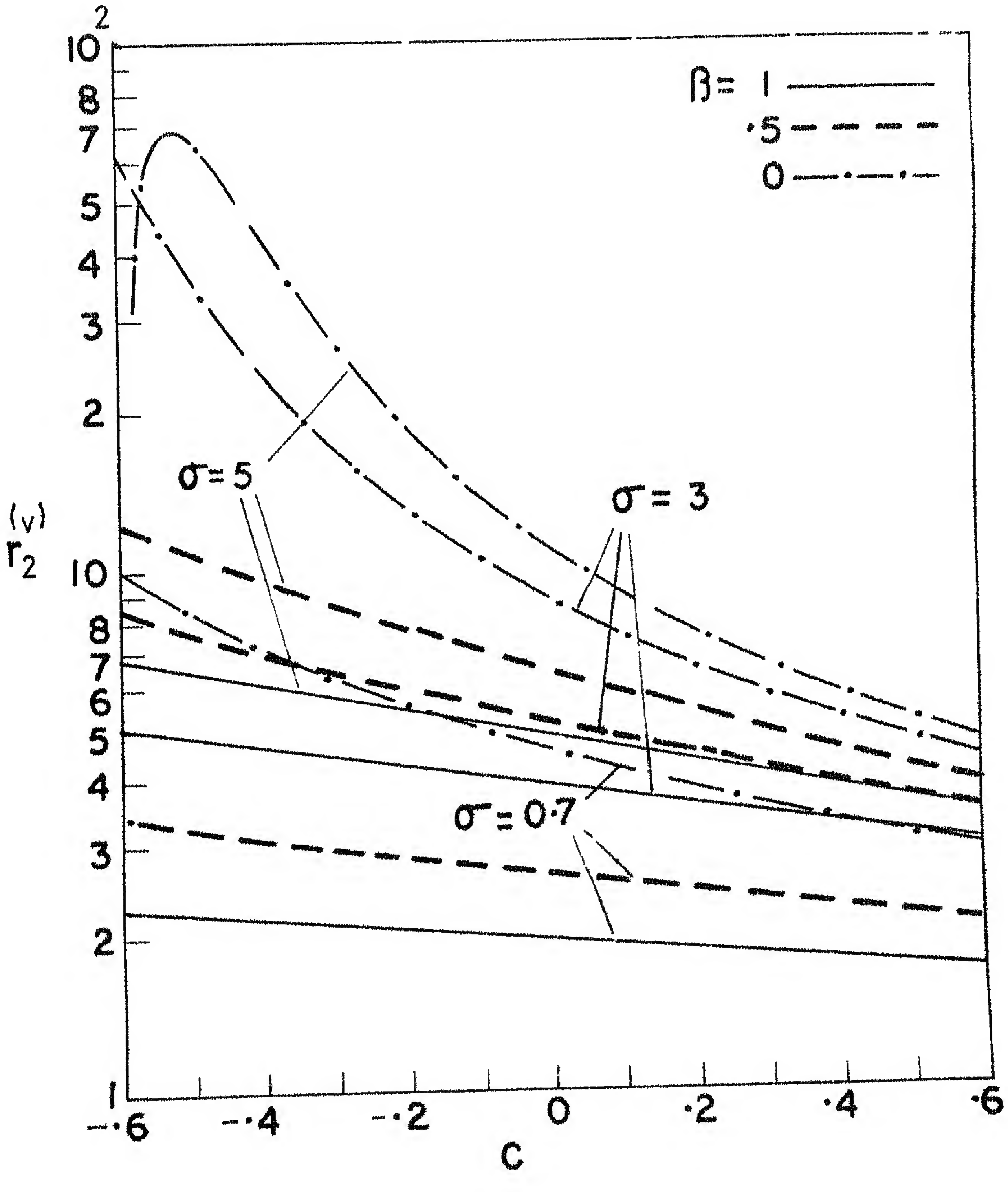

FIG 8 C EXTERNAL VORTICITY SOLUTIONS EFFECTS OF SUCTION AND INJECTION ON RECOVERY FACTOR

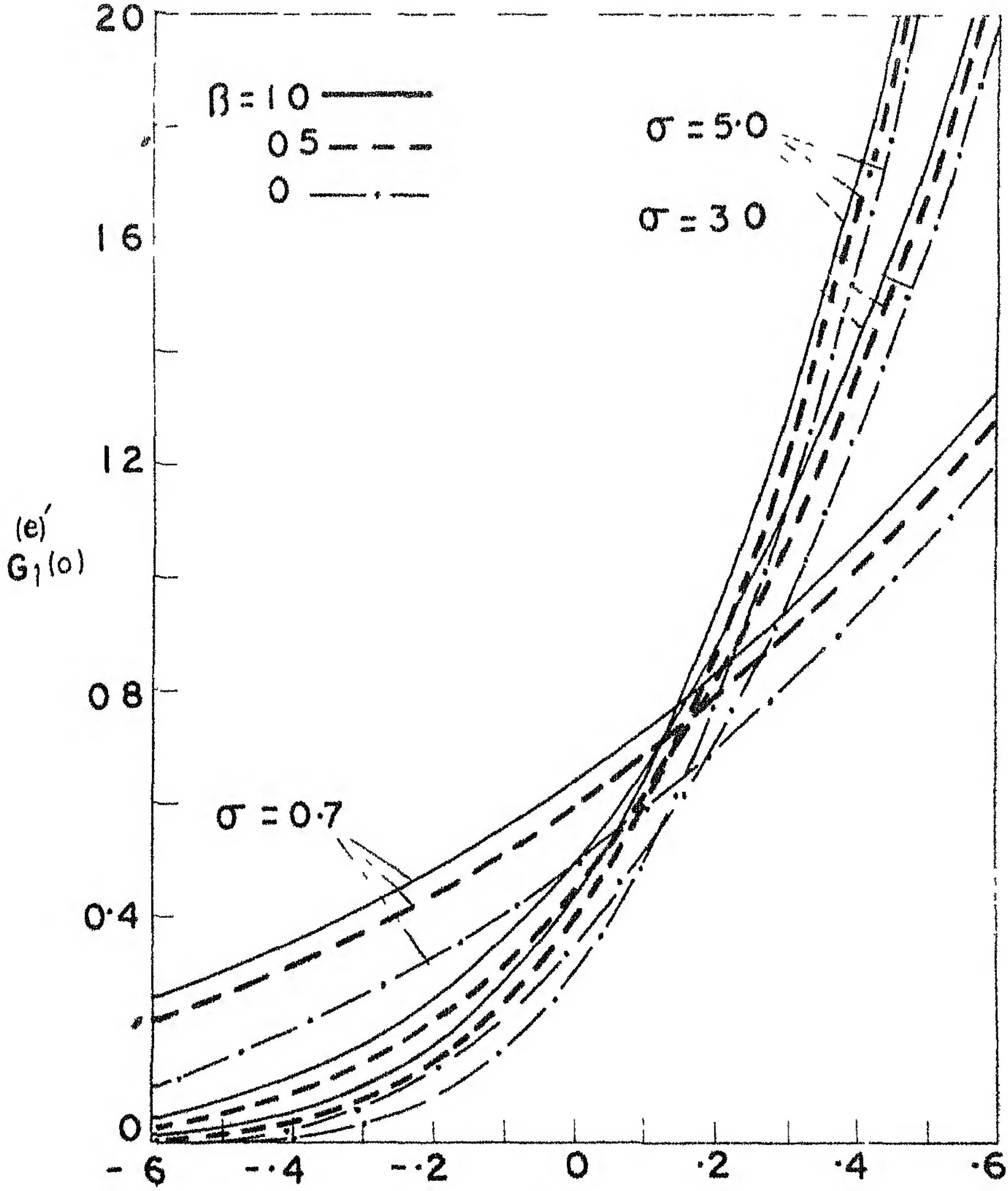

FIG 9a TEMPERATURE GRADIENT SOLUTIONS EFFECTS OF SUCTION AND INJECTION ON HEAT TRANSFER

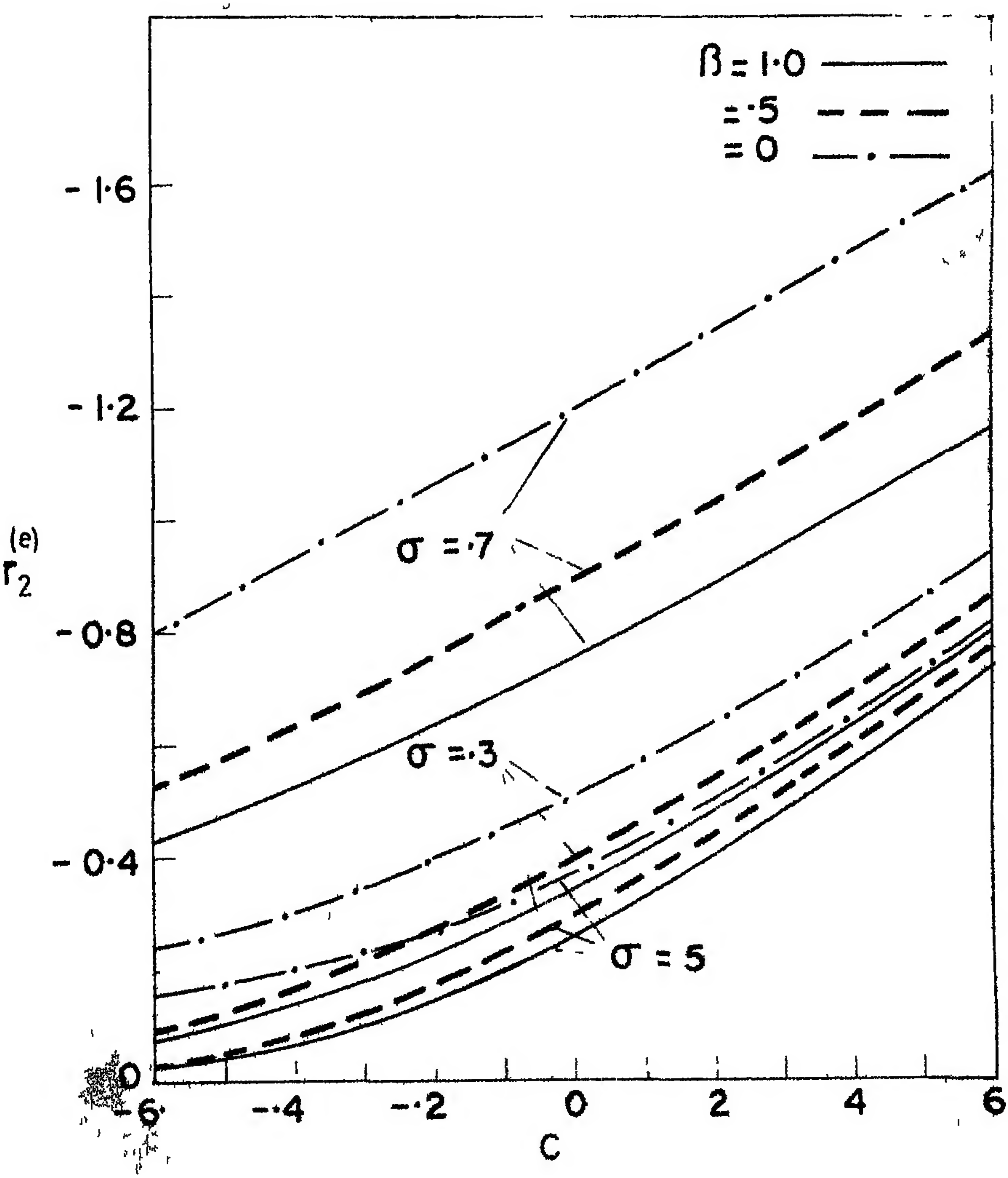

FIG. 9b TEMPERATURE GRADIENT SOLUTIONS: EFFECTS OF SUCTION AND INJECTION ON RECOVERY FACTOR

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